

# Some Gasdynamic Problems in the Flow of Condensing Vapors<sup>1</sup>

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**Abstract**—Some Gasdynamic Problems in the Flow of Condensing Vapors. The general problem of the flow of a wet vapor, with or without an inert diluent is formulated under the assumption that the liquid phase is finely divided and dispersed throughout the gaseous component in droplets whose radii are nearly constant in any local region. The processes of momentum transfer, heat transfer between phases are assumed to take place according to Stokes law and Nusselt number of unity, respectively. The mass transfer process is treated as diffusion governed in the presence of an inert diluent and kinetic governed for two phases of a pure substance.

The physical understanding of such problems, in contrast with those of conventional gas dynamics, rests largely in the role played by the relaxation times or equilibration lengths associated with these three processes. Consequently, both simple and coupled relaxation processes are examined rather carefully by specific examples. Subsequently, the problem of near-equilibrium flow in a nozzle with phase change is solved under the small-slip approximation. The structure of the normal shock in a pure substance is investigated and reveals three rather distinct zones: the gasdynamic shock, the vapor relaxation zone, and the thermal and velocity equilibration zone. The three-dimensional steady flow of the two-phase condensing continuum is formulated according to first order perturbation theory, and the structure of waves in such supersonic flow is examined. Finally, the attenuation of sound in fogs is formulated and solved accounting for the important effects of phase change as well as the viscous damping and heat transfer which have been included in previous analyses.

*Translated abstracts appear at the end of this article.*

Nomenclature			
$x_1, x_2, x_3; x, y, z$	Cartesian coordinates;	$n$	number of particles (per unit volume);
$t$	time;	$\Sigma$	droplet radius ratio, $\sigma/\sigma_0$ , local-to-reference value;
$u_1, u_2, u_3; u, v, w$	Cartesian velocity components;	$c_p, c_v$	specific heats of gas;
$m$	particle mass;	$c$	specific heat of liquid;
$\sigma$	particle radius;	$\gamma$	ratio of specific heats, $c_p/c_v$ ;
$\mu$	gas viscosity;	$e$	internal energy (per unit mass);
$\mu_v$	mass rate of vapor production;	$h$	enthalpy (per unit mass);
$\rho$	mass density;	$h_l$	latent heat of vaporization (per unit mass);
$k$	thermal conductivity, fluid;	$\kappa$	mass fraction, expressed as fraction of total gaseous component;
$p$	pressure;	$D$	binary diffusion coefficient;
$T$	temperature;	$q$	rate of heat conduction;
$a$	speed of sound;	$R$	gas constant;
$\tau$	characteristic relaxation time;	$\dot{m}$	mass flow rate through nozzle cross-section;
$\tau_{ij}$	viscous stress tensor;	$A$	cross-sectional area of nozzle;
$\lambda$	characteristic relaxation length;	$\eta$	latent heat parameter, $h_l/c_p T$ ;
		$L$	length of nozzle;
		$M$	Mach number, $u/a$ ;
		$\phi$	velocity potential, $u_i = \partial\phi/\partial x_i$ ;

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$A$	composite relaxation length, see Eq. (6.16);
$\mathcal{T}$	composite relaxation time, see Eq. (7.15);
$K = K_1 + iK_2$	propagation parameter of plane sound waves, see Eq. (7.18);
$\Omega$	composite angular velocity.
<i>Subscripts</i>	
$p$	droplet or particle;
$l$	liquid;
$v$	velocity ( $\tau_v \Rightarrow$ velocity equilibration time);
$T$	temperature ( $\tau_T \Rightarrow$ temperature equilibration time);
$D$	diffusion ( $\tau_D \Rightarrow$ vapor pressure equilibration time, diffusion limited);
$R$	kinetic ( $\tau_R \Rightarrow$ vapor pressure equilibration time, kinetic limited);
$c$	chamber, stagnation reference state (section 4) or intermediate reference state (Fig. 1);
0, 1, 2, 3	reference planes in normal shock wave structure.

### 1. Introduction

NUMEROUS scientific problems and technological situations concern the flow of vapor-liquid mixture and in most of these examples the phase change processes are of significance. Probably the most familiar are the formation of fog and rain droplets, the flow of wet steam in turbines [1, 2], nozzles [3] and condensers, and the formation of condensed phases in high-speed wind tunnels. In some instances, where the vapor expands from a dry state to a supersaturated or wet state, the question of nucleation [4, 5] is an important or even dominant one. In other cases, the droplets are already formed when the mixture enters the field of interest; then the most important physical effects to the flow field concern the exchange of mass between the phases in addition to the more conventional problems of two phase flow. It is toward this latter category of physical problem that the present work is directed. In such problems, it is the question of relaxation of the droplet motion relative to the vapor, the differences of temperature between droplets and vapor, and departure from equilibrium of the two phases that differentiates the flow of the vapor-liquid mixture from that of a simple gas.

For when the flow process in some sense takes place very slowly, the departure of the system from dynamic and thermodynamic equilibrium is very small, and the gasdynamic processes correspond to those of a simple gas of molecular weight somewhat larger than that of the vapor and satisfying an unusual equation of state. On the other hand, when the flow process takes place very rapidly, all changes in the vapor state are complete before any of the exchange processes can affect the droplet state. Thus, the mechanics and thermodynamics of the vapor and liquid phases are uncoupled. Under these circumstances, again the vapor flow behaves like that of a simple perfect gas.

For changes of state for which the characteristic time of the flow is of the same order as the relaxation or equilibration times of the transfer processes, the motion and thermodynamic changes of the vapor and liquid phase are strongly coupled. The solution then becomes much more complicated than the equilibrium or frozen flow processes, which were described above as limiting cases of very short or very long relaxation times. There are numerous processes of technological interest for which this non-equilibrium flow process is of great significance.

One of the most convenient manners of examining these relaxation and equilibration processes is through the analysis of wave motion. Here, the details of the relaxation mechanism are resolved in space or as phase lags in the response to periodic oscillations. As a consequence, after examining the physical nature of the exchange processes which are of importance in liquid-vapor flow, weak spatial waves, acoustic periodic waves and the structure of shock waves will be examined in some detail. These examples will serve to illustrate the manner in which the non-equilibrium flow processes enter into the analysis of two-phase flow problems involving phase exchange.

### 2. Interaction Processes Between Phases

Consider a dispersion of small spherical liquid droplets in a homogeneous vapor, the radii  $\sigma$  of all droplets in the neighborhood of any point being essentially the same. It will be assumed that the mass fraction of liquid and the relative densities of liquid and vapor are such that the volume occupied by the liquid is negligible. For most substances of interest, this permits the mass fraction of liquid to be as great as 10 without impairing the accuracy. Consideration will be limited to those problems where the individual droplets retain their identity or vanish, but where no new droplets are formed. Moreover, it is assumed

that the droplets will not be agglomerated or fractionated through collision or instability associated with the vapor flow over the droplets.

Under these assumptions, the coupling of dynamic and thermodynamic processes in the two phases consists of the transfer of momentum, heat, and mass between droplets and vapor. If the Reynolds number of the droplets is small enough that the Stokes drag law is applicable, then the force acting on a single droplet is

$$\varphi = 6\pi\sigma\mu(u - u_p). \quad (2.1)$$

We treat this relationship as if it were unaltered by mass and heat exchange; the mass exchange itself implies an additional momentum transfer between phases which we treat separately from that associated with viscous stresses. Now if the droplet has a mass  $m$ , the characteristic time scale with which the droplet velocity approaches that of the gas is

$$\tau_v = \frac{m}{6\pi\sigma\mu}. \quad (2.2)$$

Since the mass  $m$  and the radius  $\sigma$  are both variable during a flow process, it will be convenient to define  $\tau_v$  utilizing some constant set of values  $m_0$  and  $\sigma_0$  of the droplet mass and radius, such as their initial values. This quantity will be denoted the velocity equilibration time. Furthermore, a velocity equilibration length  $\lambda_v$  may be defined as  $\tau_v u_0$  where  $u_0$  is an appropriate characteristic velocity of the flow. Therefore,

$$\lambda_v = \frac{m_0 u_0}{6\pi\sigma_0\mu}. \quad (2.3)$$

In some instances, it is convenient to take  $u_0$  as the local sonic velocity  $a$ , for then the ratio  $a/\mu = a_0/\mu_0$  and  $\lambda_v$  are constant over the whole flow field. Physically the values of  $\tau_v$  and  $\lambda_v$  have the meaning of the time elapsed and the distance traveled, respectively, while the relative velocity of the droplet with respect to the vapor is reduced to  $e^{-1}$  of its initial value.

These facts may be used to express the force of interaction per unit volume between phases in a convenient and physically significant manner. If  $n_p$  is the number of liquid droplets per unit volume, then this force we seek may be written  $n_p\varphi$ . Also, the product  $n_p \cdot m$  is the mass of liquid phase per unit volume of space, a quantity we designate  $\rho_p$ . The total force  $F_p$  is then written

$$F_p \equiv n_p\varphi = \frac{\rho_p u_0 (u - u_p)}{\lambda_v \Sigma^2}, \quad (2.4)$$

a form which stresses the dimensional similarity

between the interphase forces and the convective transport of momentum. The quantity  $\Sigma^2$  in the denominator,  $\Sigma = \sigma/\sigma_0$ , accounts for the fact that the droplet radius, which enters into the calculations droplet mass and droplet drag force, varies from the reference value  $\sigma_0$  upon which the characteristic time and length have been based. Thus, the ratio  $\Sigma$  of local to reference droplet radii enters as a dependent variable in the problem. This feature will appear in the formulation of all interaction processes between the phases.

There is, of course, an additional momentum term that is associated with the mass transfer. As a result of condensation or vaporization at the droplet surface, mass is transferred between the velocities of the bulk liquid and bulk vapor. We have assumed that the liquid comprising a droplet is all moving at the same speed and that the shearing stresses applied to its boundary do not produce circulatory motions within the droplet. Within this approximation, the mass exchanged at the interface between the two phases has the velocity of the liquid phase. Therefore, if we designate by  $\mu_v$  the mass production rate of vapor per unit volume, then momentum is being added locally to the vapor phase at a rate

$$\mu_v u_p \quad (2.5)$$

and is being lost from the liquid phase at the same rate.

The processes of energy transfer may be handled in a manner analogous to that employed to treat momentum. Again, under the assumption of low Reynolds number, the heat transferred to a single droplet by thermal conduction is

$$q = 4\pi\sigma^2 \cdot \frac{k}{\sigma} \cdot (T - T_p) \quad (2.6)$$

where  $k$  is the thermal conductivity of the vapor and the temperature within the droplet is assumed uniform. A characteristic time

$$\tau_T = \frac{m_0 c_p}{4\pi\sigma_0 k} \quad (2.7)$$

and a characteristic length

$$\lambda_T = \frac{m_0 c_p u_0}{4\pi\sigma_0 k} \quad (2.8)$$

are readily defined. They possess the physical significance of the time elapsed and distance covered by a droplet while its temperature difference with respect to vapor is reduced to  $e^{-1}$  of its initial value. Likewise, the total heat  $Q_p$  per unit volume transferred from the vapor to liquid phases by thermal conduction is

$$Q_p \equiv n_p q = \frac{\rho_p u_0 c_p (T - T_p)}{\Sigma^2 \lambda_T}. \quad (2.9)$$

For a vapor Prandtl number of  $\frac{2}{3}$ , the velocity and thermal equilibration lengths are equal.

Simultaneously with this process, there is a transfer of heat between phases associated with the mass exchange. As mass leaves the surface of the droplet, in vapor form, it carries with it the usual internal energy of the vapor at the droplet temperature, which is in this case equal to the saturation temperature. At the same time, we see that work is being done on the vapor envelope by flux of vapor across the control surface with a normal stress equal to  $p$ , the local pressure of the vapor. Considering the surfaces of all droplets within a unit volume, the product of vapor velocity at the interface and the total surface area is  $\mu_v/\rho$ , where  $\rho$  is the density of vapor at the saturation state. The total work performed upon the vapor by this flow process is then just  $\mu_v(p/\rho)$ . The sum of this and the internal energy  $e$  carried by the vapor is

$$\mu_v \left( e + \frac{p}{\rho} \right) = \mu_v h(T_p) \quad (2.10)$$

where  $h(T_p)$  is the thermodynamic enthalpy of the vapor calculated at the droplet temperature. This equation also represents the heat lost per unit time by the droplets contained in a unit volume of space, associated with the transfer of mass between the phases. It may also be noted here that  $h(T_p)$  is equal to

$$h_p + h_v \quad (2.11)$$

the sum of the enthalpy of the liquid comprising the droplets and the latent heat of vaporization.

The third quantity which couples the two phases, and enters into the momentum and energy transfer as well, is the mass exchange rate  $\mu_v$  between the phases. In detail, there are two processes that enter into the rate at which vapor condenses on or vaporizes from a droplet. These are: (i) the kinetic rate of impact and absorption of vapor molecules upon the liquid; and (ii) the rate of diffusion of vapor from the bulk of vapor to the neighborhood of the droplet.

As a consequence, we shall consider two cases separated physically according to whether the rate of kinetic flux or diffusive flux is dominant. In general, if the system contains an inert or non-condensing gas, the vapor must diffuse through this gas in order to condense on the liquid. For any significant fraction of inert gas, the diffusion process is much slower than the kinetic process, and hence is rate controlling. The details of the kinetics may then be omitted. On the other hand, when the inert diluent is completely absent, the diffusion process vanishes also, and the rate is controlled by condensation kinetics. In the following, we shall differentiate between these two

accordingly as a diluent is present or the medium is chemically pure.

When an inert diluent is present, the mass condensation (or vaporization) rate is slow in comparison with the mass rate of collision of vapor molecules upon the droplet surface. As a consequence, the vapor and its liquid must be in thermodynamic equilibrium close to the droplet surface. Thus, the local partial pressure of the vapor is the saturation pressure corresponding to the temperature of the droplet. Under these restrictions, it is easily shown that the rate of vapor production per unit volume,  $\mu_v$ , is given by

$$\mu_v = n \cdot \rho D \cdot 4\pi\sigma \{ \kappa_v^e(T_p) - \kappa_v \} \quad (2.12)$$

where

$$\kappa_v = \rho_v/\rho = p_v/p \quad (2.13)$$

is the ratio of vapor density to total gas density and  $\kappa_v^e(T_p)$  signifies the saturated vapor concentration at the droplet temperature. The quantity  $D$  is the usual binary diffusion coefficient for vapor in the inert diluent.

In a manner similar to that we have employed for momentum and energy transfer, a characteristic time and length may be introduced for this process

$$\tau_D = \frac{m_0}{4\pi\sigma_0(\rho D)\Sigma^2} \quad (2.14)$$

$$\lambda_D = \frac{m_0 u_0}{4\pi\sigma_0(\rho D)\Sigma^2} \quad (2.15)$$

and the physical interpretations of each are similar to those that have been given previously. Because the product of gas density and binary diffusion coefficient is very nearly equal to the coefficient of viscosity,  $\mu$  for the gas, that is the Schmidt number  $\mu/\rho D$  is nearly unity, this characteristic length is nearly equal to the velocity and thermal equilibration lengths. The vapor mass production may then be written

$$\mu_v = \frac{\rho_p u_0}{\Sigma^2 \lambda_D} \{ \kappa_v^e(T_p) - \kappa_v \} \quad (2.16)$$

where diffusion of vapor through inert diluent is the rate controlling process.

When we are dealing with a pure substance, however, the condensation flux rates take place at nearly molecular velocities. At present, the formulation of this vapor transfer process is incomplete; the only conclusive result is that of Hertz [6] and Knudsen [7], valid in the free molecular flow regime. It is based on the assumption that molecular flow from the liquid may be computed using a Maxwellian distribution at liquid droplet temperature  $T_p$  and number density corresponding to saturation pressure  $p^e(T_p)$ .

Similarly, the flux of molecules from the surrounding vapor surface is computed from a Maxwellian distribution at the vapor state  $p$ ,  $T$ . Utilizing the well-known Maxwellian mass-flux rate, the net condensation rate per unit area is

$$\frac{p}{\sqrt{(2\pi RT)}} - \frac{p^e(T_p)}{\sqrt{(2\pi RT_p)}} \approx \frac{1}{2\pi} \rho \sqrt{(2\pi RT)} \times \left\{ \frac{p - p^e}{p} - \frac{1}{2} \left( \frac{T - T_p}{T} \right) \right\} \quad (2.17)$$

where the approximation holds for small differences between liquid and vapor temperatures. The free-molecule restriction follows from the tacit assumption that the molecular streams from the vapor and from the liquid surface may move independently.

On the other hand, a rough approximation to condensate flux from a continuum flow may be made in somewhat the same spirit as that employed by Bolza *et al.* [8] in treating the temperature jump at a solid surface. To carry this out, suppose that while the molecular mass flux from the liquid takes place according to the kinetic flux from the liquid state  $p^e(T_p)/\sqrt{(2\pi RT_p)}$ , the flux from the vapor to the surface takes place from some unknown intermediate state  $p_i T_i$ . This mass is written  $p_i/\sqrt{(2\pi RT_i)}$ . Referring to Fig. 1, these fluxes couple with the equilibrium

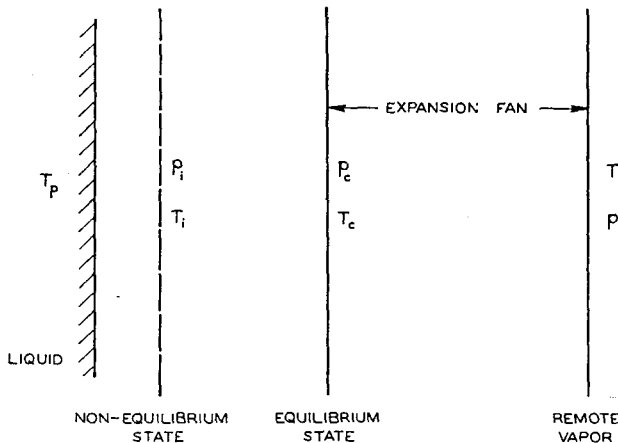


FIG. 1. Simple model for condensation on a plane surface.

vapor state  $p_c T_c$  several mean free paths distant. This coupling occurs through the conservation of mass

$$\frac{p_i}{\sqrt{(2\pi RT_i)}} - \frac{p^e(T_p)}{\sqrt{(2\pi RT_p)}} = -\rho_c u_c, \quad (2.18)$$

the conservation of momentum

$$\frac{1}{2} p_i + \frac{1}{2} p^e(T_p) = p_c + \rho_c u_c^2, \quad (2.19)$$

and the conservation of energy

$$\begin{aligned} \frac{1}{\pi} \sqrt{(2\pi RT_i)} \cdot p_i - \frac{1}{\pi} \sqrt{(2\pi RT_p)} \cdot p^e(T_p) \\ = -\rho_c u_c (c_p T_c + \frac{1}{2} u_c^2). \end{aligned} \quad (2.20)$$

These relations determine the intermediate state as well as the mass flow; for small differences between liquid and vapor temperature, the mass flow is

$$-\rho_c u_c = \frac{1}{2\pi} \frac{8}{2 + [\gamma/(\gamma - 1)]} \rho_c \sqrt{(2\pi RT_c)} \left( \frac{p_c - p^e(T_p)}{p_c} \right). \quad (2.21)$$

The continuum state is not, however, the state of the remote vapor because the continuum state is moving toward the surface and the remote vapor is stationary at the state  $p$ ,  $T$ . These two states are connected through an isentropic expansion fan which relates the pressure difference to the mass flow as

$$\frac{p_c - p}{p} \approx \gamma \frac{\rho_c u_c}{\rho a}, \quad (2.22)$$

where  $a = \sqrt{(\gamma RT)}$  is the sound speed in the remote vapor. For small state variations, then, the mass flux per unit area may be written

$$\begin{aligned} \frac{1}{1 + \sqrt{\left(\frac{2\pi}{\gamma}\right) \left(\frac{3\gamma + 2}{8}\right)}} \cdot \frac{1}{\gamma} \rho a \left( \frac{p - p^e(T_p)}{p} \right) \\ \cong 0.23 \rho a \left( \frac{p - p^e(T_p)}{p} \right) \end{aligned} \quad (2.23)$$

where  $\gamma = \frac{5}{3}$ . It is notable that the mass flux does not depend upon the temperature difference as it does for free molecule flow, but only upon the effective pressure difference between the phases. Moreover, the velocity at which material is being transferred from one phase to the other is a substantial fraction of the bulk sound velocity, and consequently the characteristic time of the process may be short compared with that for diffusive exchange.

Recently, in his doctoral thesis [9], P. N. Shankar has examined this exchange process from a detailed kinetic theory viewpoint utilizing a moment method for the approximate solution of the Boltzmann equation for both plane and spherical examples. For continuum flow, that is, where the ratio of mean free path to droplet radius is small, he finds that the result is of the form (2.23), but with the constant equal to 0.22 instead of 0.23. For free molecule flow, where the ratio of mean free path to droplet radius is large, the Hertz-Knudsen formula reappears. The transition from free-molecule to continuum is to suppress the dependence of the transfer ratio upon the temperature difference between phases.

We shall restrict our considerations here to the continuum range and take the mass exchange rate between phases as

$$\alpha \rho a \left( \frac{p - p^e(T_p)}{p} \right).$$

where  $\alpha$  is a constant of order unity. Then the mass rate of vapor production per unit volume of mixture is

$$\mu_v = -n \cdot 4\pi\sigma^2 \cdot \alpha \rho a \left( \frac{p - p^e}{p} \right), \quad (2.24)$$

and for this process we shall denote the saturation relaxation time

$$\tau_R = \frac{m_0}{4\pi\sigma_0\rho(\alpha\sigma_0a)}, \quad (2.25)$$

and the corresponding relaxation length

$$\lambda_R = \frac{m_0 u_0}{4\pi\sigma_0\rho(\alpha\sigma_0a)} = \frac{1}{3\alpha} \frac{u_0}{a} \frac{\rho_l}{\rho} \sigma_0. \quad (2.26)$$

The relationship between these quantities and the corresponding quantities for the diffusion governed process, Eqs. (2.14) and (2.15), is interesting. As they have been written, it is clear that

$$\frac{\tau_R}{\tau_D} = \frac{D}{\alpha\sigma_0a} \sim \frac{\lambda}{\sigma_0}, \quad (2.27)$$

using the elementary kinetic theory expression for the diffusion coefficient. Therefore, it appears that  $\tau_R/\tau_D \ll 1$ , provided that the mean free path in the vapor is small in comparison with the droplet radius. But this must be true for the continuum form of the mass-transfer law (2.23) to be valid. Thus, we may conclude that for our discussion,

$$\tau_R/\tau_D \ll 1,$$

so that the "saturation relaxation" process in a

chemically pure substance is fast in comparison with all of the other relaxation processes.

It should be pointed out, although we shall not develop the subject along these lines, that when the droplets are sufficiently large or the slip velocities sufficiently large, the processes that define  $\lambda_v$ ,  $\lambda_T$ , and  $\lambda_D$  depend quadratically upon the "driving potentials" rather than linearly. If the drag of a particle, for example, is in the quadratic range with drag coefficient  $C_D$ , then the drag force per unit volume is

$$n \cdot \pi\sigma^2 \cdot \rho \frac{(u - u_p)^2}{2} C_D.$$

The velocity equilibration length, defined in the manner used previously, gives

$$\lambda_v = \frac{2m_0 C_D}{\pi\sigma_0^2 \rho} = \frac{8}{3} C_D \frac{\rho_l}{\rho} \sigma_0. \quad (2.28)$$

The values of  $\lambda_T$  and  $\lambda_D$  have similar magnitudes. For high speed flows, these are the same general order as  $\lambda_v$ , and consequently the peculiar nature of the saturation relaxation process vanishes.

Table 1 gives values for these characteristic lengths for one micron droplets in potassium vapor at two saturation states.

### 3. Continuum-Like Conservation Relations

The interaction processes that have been described enter into and modify the governing conservation equations for the two-phase flow process. We shall not derive these various relations or dwell upon the details of approximations that permit treating the droplet cloud as a continuum. The details of this general technique applicable to obtaining these equations are given in Ref. 10. Rather, we shall stress only such points as make the physical processes clear.

Table 1. Characteristic Lengths for Wet Potassium Vapor ( $\sigma = 1.0 \mu$ ,  $u_0 = a$ )

Length	Representative expression	$T = 2000^\circ\text{R}$ $p = 29 \text{ psi}$	$T = 2500^\circ\text{R}$ $p = 165 \text{ psi}$
$\left. \begin{matrix} \lambda_v \\ \lambda_T \\ \lambda_D \end{matrix} \right\} \begin{matrix} \text{linear} \\ \text{law} \end{matrix}$	$\frac{2}{9} \frac{\rho_l \sigma^2 a}{\mu}$	$4.23 \times 10^{-1} \text{ cm}$	$3.78 \times 10^{-1} \text{ cm}$
$\lambda_R$	$\frac{1}{3} \frac{\rho_l}{\rho} \sigma$	$2.26 \times 10^{-2} \text{ cm}$	$4.68 \times 10^{-3} \text{ cm}$
$\left. \begin{matrix} \lambda_v \\ \lambda_T \\ \lambda_D \end{matrix} \right\} \begin{matrix} \text{quadratic} \\ \text{law} \end{matrix}$	$\frac{8}{3} \frac{\rho_l}{\rho} \sigma$	$1.71 \times 10^{-1} \text{ cm}$	$3.75 \times 10^{-2} \text{ cm}$

Consider a chemically pure substance consisting of a vapor and its liquid phase, the latter existing in the form of small droplets that are locally of a single size. These droplets shall be considered sufficiently numerous so that a large number exist in a volume of characteristic dimensions corresponding to the smallest distance of interest in the problem but sufficiently spaced that the individual droplets are non-interacting in the sense of Ref. 10. Moreover, we shall assume that droplets may grow or diminish in size as a result of condensation or vaporization but that no nucleation, coalescence, or shattering takes place during the flow process. Thus, a droplet has an identity that persists with the possible exception of vanishing due to complete vaporization.

Then the density  $\rho$  of the vapor and the corresponding velocity vector  $u_i$  are related in the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = \mu_v \quad (3.1)$$

to the local mass production rate of vapor  $\mu_v$ . Similarly, the effective particle density  $\rho_p$ , mass per unit volume in the liquid state, is related to the velocity vector  $u_{pi}$  of the particle cloud as

$$\frac{\partial \rho_p}{\partial t} + \frac{\partial}{\partial x_j}(\rho_p u_{pj}) = -\mu_v \quad (3.2)$$

because the mass production rate of liquid phase must be equal and opposite that of the vapor.

The equations of motion of the gas phase differ from the conventional form in that (i) an effective volumetric force  $-F_{pi}$  exists representing the force exerted by the particle cloud on the vapor as the particles move through it, and (ii) the appearance or disappearance of vapor, and hence momentum, at the droplet speed. The droplet force is represented in the form developed in Section 2:

$$-F_{pi} = \frac{\rho_p a(u_{pi} - u_i)}{\Sigma^2 \lambda_v} \quad (3.3)$$

The momentum added to the vapor phase locally is just the product of the vapor mass  $\mu_v$  generated locally per unit time and the local particle velocity  $u_{pi}$  from which the vapor flows or is absorbed. That is, the momentum per unit volume added to the vapor phase because of phase change is

$$\mu_p u_{pi} \quad (3.4)$$

The momentum equations for the vapor phase only are then

$$\begin{aligned} \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_j u_i) &= \frac{\rho_p a(u_{pi} - u_i)}{\Sigma^2 \lambda_v} \\ &\quad - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \mu_v u_{pi}, \end{aligned} \quad (3.5)$$

where  $\tau_{ij}$  is the viscous stress tensor appropriate to the vapor alone.

For the liquid phase, the forces acting per unit volume are equal and opposite to those acting on the vapor, and momentum gained by the vapor phase resulting from mass transfer must appear as a vapor loss in the liquid phase. The equation of motion governing the motion of the particle cloud is therefore

$$\begin{aligned} \frac{\partial}{\partial t}(\rho_p u_{pi}) + \frac{\partial}{\partial x_j}(\rho_p u_{pj} u_{pi}) \\ = - \frac{\rho_p a(u_{pi} - u_i)}{\Sigma^2 \lambda_v} - \mu_v u_{pi}, \end{aligned} \quad (3.6)$$

the drag term associated with the gross pressure gradient  $\partial p / \partial x_i$  being negligible for very small droplets.

To complete the formal statement of the conservation laws, we apply the conservation of energy, the first law of thermodynamics, to the separate systems. The energy per unit volume  $\rho(e + \frac{1}{2}u_k u_k)$  of the vapor is changed by (i) the rate at which work is done on the system by external forces, (ii) the rate at which heat is transferred to the vapor, and (iii) the gain of energy from the vapor mass increase due to phase change. Work is done by the surface stresses  $-p\delta_{ij} + \tau_{ij}$  and the particle forces  $-F_{pi}$ . These represent a work per unit volume of

$$\frac{\partial}{\partial x_j} \{(-p\delta_{ij} + \tau_{ij})u_i\} - u_{pi}F_{pi} \quad (3.7)$$

since the particles move with the velocity  $u_{pi}$ . If we define the heat flux vector  $q_i$  as the heat flowing out through a unit area of surface per unit time, the heat lost per unit volume is  $\partial q_i / \partial x_i$ . On the other hand, the heat  $-Q_p$  added per unit volume of vapor from the liquid phase has been calculated in Section 2 as

$$-Q_p = \frac{\rho_p a c_p (T_p - T)}{\Sigma^2 \lambda_T} \quad (3.8)$$

Finally, it remains to account for the effects of mass transfer upon the total energy of the vapor phase. Because the vapor flowing from the droplets carries its own internal energy and does work as it moves across the control surface surrounding the vapor droplet interface, it contributes, as was shown in Section 2, the flux of enthalpy  $\mu_v h(T_p)$  evaluated at the droplet temperature. In addition, of course, this vapor carries a kinetic energy per unit mass computed from

the droplet velocity. The energy associated with mass transfer to the vapor phase is therefore

$$\mu_v \{ (T_p) + \frac{1}{2} u_{pk} u_{pk} \}. \quad (3.9)$$

The energy equation for the vapor phase is then

$$\begin{aligned} \frac{\partial}{\partial t} \{ \rho (e + \frac{1}{2} u_k u_k) \} + \frac{\partial}{\partial x_j} \{ \rho u_j (e + \frac{1}{2} u_k u_k) \} \\ = - \frac{\partial}{\partial x_j} (u_j p) + \frac{\partial}{\partial x_j} (\tau_{ij} u_i) \\ + \frac{\rho_p a u_{pi} (u_{pi} - u_i)}{\Sigma^2 \lambda_v} + \frac{\rho_p a c_p (T_p - T)}{\Sigma^2 \lambda_T} - \frac{\partial q_j}{\partial x_j} \\ + \mu_v \{ h(T_p) + \frac{1}{2} u_{pk} u_{pk} \}. \end{aligned} \quad (3.10)$$

With the aid of the equations of continuity, motion, and some manipulation, the energy equation for the vapor phase may be written in the form

$$\begin{aligned} \rho \frac{\partial e}{\partial t} + \rho u_j \frac{\partial e}{\partial x_j} = \left\{ \frac{\mu_v}{\rho} - \frac{\partial u_j}{\partial x_j} \right\} p \\ + \left\{ \frac{\rho_p a c_p (T_p - T)}{\Sigma^2 \lambda_T} + \mu_v (e(T_p) - e(T)) - \frac{\partial q_j}{\partial x_j} \right\} \\ + \tau_{ij} \frac{\partial u_i}{\partial x_j} + \frac{\rho_p a (u_{pi} - \dot{u}_i)^2}{\Sigma^2 \lambda_v}. \end{aligned} \quad (3.11)$$

The terms in curly brackets signify, in order of their occurrence, (i) the work done by pressure in changing the volume, (ii) the heat transferred to the vapor phase, and (iii) the dissipation due to viscous stresses within the vapor and to droplets moving through the vapor. The unusual terms are  $\mu_v/\rho$ , the rate of flow over the surfaces bounding the droplets, and  $\mu_v[e(T_p) - e(T)]$ , the internal energy carried across the same boundary.

The energy equation for the droplet cloud may be written down directly from the foregoing consideration

$$\begin{aligned} \frac{\partial}{\partial t} \{ \rho_p (e_p + \frac{1}{2} u_{pk} u_{pk}) \} + \frac{\partial}{\partial x_j} \{ \rho_p u_{pj} (e_p + \frac{1}{2} u_{pk} u_{pk}) \} \\ = - \frac{\rho_p a u_{pi} (u_{pi} - u_i)}{\Sigma^2 \lambda_v} - \frac{\rho_p a c_p (T_p - T)}{\Sigma^2 \lambda_T} \\ - \mu_v \{ e_p + h_l + \frac{1}{2} u_{pk} u_{pk} \}, \end{aligned} \quad (3.12)$$

where  $h_l$  is the latent heat of vaporization of the droplets. Again utilizing the equations of motion and continuity for the droplet cloud, the energy equation may be written

$$\rho_p \frac{\partial e_p}{\partial t} + \rho_p u_{pj} \frac{\partial e_p}{\partial x_j} = - \frac{\rho_p a c_p (T_p - T)}{\Sigma^2 \lambda_T} - \mu_v h_l. \quad (3.13)$$

This has the simple interpretation that in following an element of the droplet cloud, the change in internal energy experienced by it is accounted for by the heat transferred away from the particles and the latent heat that must be supplied by the particles to create the vapor that is produced.

Another useful form for the total conservation of energy may be obtained through summing Eqs. (3.10) and (3.12) and taking account of the thermodynamic relationship

$$h(T_p) = c_p(T_p) + h_l. \quad (3.14)$$

This form then becomes

$$\begin{aligned} \frac{\partial}{\partial t} \{ \rho (e + \frac{1}{2} u_k u_k) + \rho_p (e_p + \frac{1}{2} u_{pk} u_{pk}) \} \\ + \frac{\partial}{\partial x_j} \{ \rho u_j (e + \frac{1}{2} u_k u_k) + \rho_p u_{pj} (e_p + \frac{1}{2} u_{pk} u_{pk}) \} \\ = - \frac{\partial}{\partial x_j} (u_j p) + \frac{\partial}{\partial x_j} (\tau_{ij} u_i) - \frac{\partial q_j}{\partial x_j}. \end{aligned} \quad (3.15)$$

The terms describing the exchange process between phases cancel, as indeed they must, when the vapor and droplets together comprise the thermodynamic system.

Following a common practice in two-phase flow analysis, we have assumed that the temperature of a droplet is uniform throughout its volume. This assumption is accurate to a degree that depends upon the droplet size and physical properties as well as upon its temperature environment. The approximation enters the problem when we employ the same temperature to describe both the droplet surface processes and the droplet enthalpy.

The physical effect of non-uniform temperature distribution is to delay the response in droplet enthalpy corresponding to a change in surface temperature. The amount of this lag, and the physical quantities that enter into it, may be determined by calculating the enthalpy  $h_p$  of a homogeneous droplet whose surface is subjected to a periodic variation in surface temperature  $T_p$ . If the liquid has mass density  $\rho_l$ , specific heat  $c$ , and thermal conductivity  $k_l$ , then the enthalpy of a droplet of radius  $\sigma$  lags approximately

$$\frac{2}{15} \frac{\omega \sigma^2 \rho_l c}{k_l}$$

radians behind a surface fluctuation of angular velocity  $\omega$ . The time lag is somewhat more convenient, and this is

$$\tau_c = \frac{2}{15} \frac{\sigma^2 \rho_l c}{k_l}.$$



Now this time lag is small for very small droplets, but to estimate its significance,  $\tau_c$  must be compared with other relaxation times in our system. The most natural standard of comparison is the thermal relaxation time  $\tau_T$  for thermal equilibration between the droplet surface and the surrounding vapor. This ratio is

$$\frac{\tau_c}{\tau_T} = \frac{2}{5} \left( \frac{c}{c_p} \right) \left( \frac{k}{k_l} \right)$$

where  $c/c_p$  is the ratio of specific heats for droplet and vapor while  $k/k_l$  is the ratio of thermal conductivities of vapor and liquid. It is significant to note that the relative importance of non-uniform temperature distribution in the droplet depends only upon the system physical properties and not upon the droplet radius. For potassium droplets in potassium vapor,  $\tau_c/\tau_T \cong 1.7 \times 10^{-4}$ , while for water droplets in air,  $\tau_c/\tau_T \cong 1.1 \times 10^{-1}$ .

Because we have restricted consideration to a chemically pure substance, the vapor mass production rate  $\mu_v$  must be given in the form of Eq. (2.24). Utilizing the characteristic length defined in (2.26),

$$\mu_v = \frac{\rho_p a}{\Sigma \lambda_R} \left( \frac{p^e(T_p) - p}{p} \right), \quad (3.16)$$

where

$$\lambda_R = \frac{1}{3\alpha} \frac{\rho_l}{\rho} \sigma_0. \quad (3.17)$$

The saturation pressure  $p^e(T_p)$  at the droplet temperature may be defined by the Clausius-Clapeyron equation

$$\frac{p^e(T_p)}{p_0} = \exp \left\{ \frac{h_l}{RT_0} \left( 1 - \frac{T_0}{T_p} \right) \right\} \quad (3.18)$$

where  $p_0$ ,  $T_0$  describe a reference saturation state. In utilizing this relationship, we assume the vapor to be an ideal gas with constant latent heat.

Finally, we must obtain a relation that determines the particle size locally. The mass of a single particle,  $\rho_l \cdot \frac{4}{3}\pi\sigma^3$ , varies due to the condensation or vaporization taking place at its surface. Now the rate at which a single droplet gains mass is, according to Eq. (2.24),  $-\mu_v/n$ , and therefore we write

$$\left( \frac{\partial}{\partial t} + u_{pj} \frac{\partial}{\partial x_j} \right) \left( \frac{4}{3}\pi\sigma^3 \rho_l \right) = 4\pi\sigma^2 \alpha \rho a \left( \frac{p - p^e(T_p)}{p} \right).$$

This may be simplified by introducing the droplet radius ratio  $\Sigma = \sigma/\sigma_0$  and the condensation relaxation length, (2.26), using the local sound speed  $a$  as the characteristic velocity. Then we have

$$\frac{\partial \Sigma}{\partial t} + u_{pj} \frac{\partial \Sigma}{\partial x_j} = \frac{a}{3\lambda_R} \left( \frac{p - p^e(T_p)}{p} \right). \quad (3.19)$$

In many calculations the flow problem may be solved without accounting for the variation of  $\Sigma$ , and the distribution of droplet size is found subsequently.

#### 4. Some One-Dimensional Problems

Some of the most interesting technological problems involving two-phase flow are of an essentially one-dimensional character. Among these are the flow through nozzles and the structure of shock waves. In this section, these two problems will be examined. In addition to the intrinsic value of investigating the effects of phase change upon these classical examples, the results will illustrate the near-equilibrium approximation and the influence of the rates of various relaxation processes on the structure of the flow fields.

##### Near-equilibrium nozzle flow

The flow through a convergent-divergent nozzle of a vapor, together with a liquid condensate cloud consisting of small droplets of a single radius, may be described as a one-dimensional problem where the area  $A(x)$  varies sufficiently slowly along the  $x$ -axis. Then utilizing the general relations developed in the preceding section, the problem is described by equations of continuity for the vapor phase

$$\frac{1}{A} \frac{d\dot{m}}{dx} = \mu_v \quad (4.1)$$

where  $\dot{m} \equiv \rho u A$  is the mass flow rate of vapor through any cross section, together with an overall mass conservation relation

$$\dot{m} + \dot{m}_p = (1 + \kappa_{pc}) \dot{m}_c \quad (4.2)$$

Here,  $\dot{m}_c$  is the initial flow rate of vapor leaving the reservoir before any phase change has taken place, and  $\kappa_{pc}$  is the ratio of liquid to vapor flow rates at the same initial state. The equation of motion for the vapor phase is

$$\rho u \frac{du}{dx} = - \frac{dp}{dx} + \frac{\rho_p a (u_p - u)}{\Sigma^2 \lambda_v} + \mu_v (u_p - u), \quad (4.3)$$

and the corresponding relation for liquid droplets is

$$\rho_p u_p \frac{du_p}{dx} = - \frac{\rho_p a (u_p - u)}{\Sigma^2 \lambda_v} \quad (4.4)$$

Finally, the first law of thermodynamics for the liquid phase may be written

$$\rho_p u_p \frac{de_p}{dx} = - \frac{\rho_p a c_p (T_p - T)}{\Sigma^2 \lambda_T} - \mu_v h_l \quad (4.5)$$

and energy conservation for the entire system is

$$\dot{m}(h + \frac{1}{2}u^2) + \dot{m}_p(e_p + \frac{1}{2}u_p^2) = \dot{m}_c h(T_c) + \kappa_{pc} \dot{m}_c e_p(T_c) \quad (4.6)$$

where  $T_c$  is the temperature of the reservoir condition, assumed to be a stagnation state. As in the more general three-dimension problem, the vapor mass production rate is given by (3.16) and the local saturation pressure corresponding to the droplet temperature is given by (3.18), the Clausius-Clapeyron relation; for the present problem, the reference state is taken as  $p_c$ ,  $T_c$ , corresponding to the stagnation condition in the reservoir.

Now if the expansion process through the nozzle took place such that dynamic and thermodynamic equilibrium existed locally, that is,  $u_p = u$ ,  $T_p = T$ ,  $p^e(T_p) = p$ , then the state of the gas at any point in the nozzle, defined by the pressure ratio  $p/p_c$ , is easily obtained

$$\frac{T_c}{T^{(0)}} = 1 + \frac{\gamma - 1}{\gamma} \frac{1}{\eta} \log \frac{p_c}{p} \quad (4.7)$$

$$\frac{(u^{(0)})^2}{2c_p T_c} = \left(1 + \frac{\eta}{1 + \kappa_{pc}}\right) \left(1 - \frac{T^{(0)}}{T_c}\right) - \frac{T^{(0)}}{T_c} \log \left(\frac{T_c}{T^{(0)}}\right) \quad (4.8)$$

$$\kappa_p^{(0)} - \kappa_{pc} = - \left(1 - \frac{T^{(0)}}{T_c}\right) - \left(\frac{1 + \kappa_{pc}}{\eta}\right) \frac{T^{(0)}}{T_c} \log \frac{T_c}{T^{(0)}} \quad (4.9)$$

The superscript "0", as in  $T^{(0)}$ , indicates the local value of this dependent variable when the equilibrium is satisfied. The quantity  $\kappa_p$  is used to denote

$$\kappa_p \equiv \frac{\dot{m}_p}{\dot{m}_c} \quad (4.10)$$

the ratio of local liquid mass flow to initial vapor flow. This ratio varies, of course, because of the exchange process between phases that occurs during the expansion. Since we consider pressure as the independent variable, the area required is easily computed from the equation of continuity and the state equation for a perfect gas

$$p = \rho^{(0)} R T^{(0)} \quad (4.11)$$

$$\rho^{(0)} u^{(0)} A^{(0)} = \dot{m}_c \kappa_v^{(0)}. \quad (4.12)$$

where  $\kappa_v$  is the ratio of local vapor flux to the initial value before phase change,

$$\kappa_v = \frac{\dot{m}}{\dot{m}_c} = 1 + \kappa_{pc} - \kappa_p. \quad (4.13)$$

We wish, however, to solve the nozzle flow problem when the phases are not in equilibrium. For small liquid droplets and nozzles of reasonable length, the departure from equilibrium is usually quite small. As a consequence, it is possible to extend the "small slip" approximation, developed extensively in Refs. 10 and 11 for velocity and temperature lag, to the nozzle problem with phase change. The general idea of this technique arises from the observation that small values of  $\lambda_v$ ,  $\lambda_T$ ,  $\lambda_R$ —small because of small droplet size—necessitate respectively that  $u_p - u$ ,  $T_p - T$ , and  $p^e - p$  must be small to insure that terms occurring in Eqs. (4.4), (4.5), and (3.16) shall be finite. More formally, it is implied that all dependent variables may be expanded in power series of the small parameter  $\lambda/L$ ,  $\lambda$  being of order  $\lambda_v$ ,  $\lambda_T$ ,  $\lambda_R$ , and  $L$  the nozzle length. In particular, it appears that the quantities  $(u_p - u)/a$ ,  $(T_p - T)/T_c$ ,  $(p^e - p)/p_c$  are of the order  $\lambda/L$ ; the quantities  $u$ ,  $T$ ,  $\kappa_p$ , etc., have terms independent of  $\lambda/L$  that are, in fact, the values  $u^{(0)}$ ,  $T^{(0)}$ ,  $\kappa_p^{(0)}$  calculated previously. Formally, then, the solution we have called the equilibrium flow is attained by letting  $\lambda/L \rightarrow 0$ .

We shall calculate here only the first order perturbation terms for the nozzle flow and, as a consequence, second and higher order terms in small quantities will be neglected as they arise. It is convenient also to introduce three new dependent variables

$$\left. \begin{aligned} u_s &\equiv u - u_p = \frac{\lambda_v}{L} u_s^{(1)} + \dots \\ T_s &\equiv T - T_p = \frac{\lambda_v}{L} T_s^{(1)} + \dots \\ p_s &\equiv p - p^e = \frac{\lambda_v}{L} p_s^{(1)} + \dots \end{aligned} \right\} \quad (4.14)$$

to replace  $u_p$ ,  $T_p$ , and  $p^e$ , respectively. Since each of these is small in the sense described, only linear terms in  $u_s$ ,  $T_s$ , and  $p_s$  shall be retained. In a corresponding manner, we shall express the other variables as perturbation series

$$\left. \begin{aligned} u &= u^{(0)} + \frac{\lambda_v}{L} u^{(1)} + \dots \\ T &= T^{(0)} + \frac{\lambda_v}{L} T^{(1)} + \dots \\ \kappa_v &= \kappa_v^{(0)} + \frac{\lambda_v}{L} \kappa_v^{(1)} + \dots \end{aligned} \right\} \quad (4.15)$$

in which only the first-order term will be retained in the calculation. In all expansions, the velocity equilibration length  $\lambda_v$  is used to indicate the order of the perturbation parameter and it is assumed that  $\lambda_v/L \ll 1$ .

Before carrying out the perturbation analysis, it is convenient to make the change of variable suggested by (4.13) and, in the process, delete the high order small quantities that arise in the process. By manipulating Eqs. (4.5) and (4.6), together with the equation of motion for each phase, we obtain

$$(1 + \kappa_{pc}) \frac{d}{dx} \left( \frac{T}{T_c} \right) - \frac{\gamma - 1}{\gamma} \kappa_v \left( \frac{\rho_c}{\rho} \right) \frac{d}{dx} \left( \frac{p_c}{p} \right) + \eta \frac{d\kappa_v}{dx} \\ \cong \frac{d}{dx} \left( \kappa_p \frac{T_s}{T_c} \right) + \frac{\kappa_p}{\Sigma^2 \lambda_v} \frac{a}{u_p c_p T_c} u_s^2. \quad (4.16)$$

In the state of local equilibrium, this reduces to the first law of thermodynamics for the mixture; it plays a corresponding role in our calculation. The pressure gradient on the left-hand side of Eq. (4.15) may be eliminated by use of the Clausius-Clapeyron equation. Differentiating Eq. (3.18) with respect to  $x$  after writing  $p^e = p - p_s$ , the resulting equation reduces to

$$\frac{d}{dx} \log \left( \frac{T}{T_c} \right) + \frac{d}{dx} \left( \eta \cdot \frac{\kappa_v}{1 + \kappa_{pc}} \cdot \frac{T_c}{T} \right) \cong \frac{\kappa_v}{1 + \kappa_{pc}} \\ \times \left\{ \frac{\gamma - 1}{\gamma} \frac{1}{p} \frac{dp}{dx} \cdot \frac{T_s}{T} + \frac{\gamma - 1}{\gamma} \frac{1}{p} \frac{dp_s}{dx} - \eta \frac{T_c}{T^2} \frac{dT_s}{dx} \right. \\ \left. - \eta \frac{T_c}{T^2} \frac{dT}{dx} \cdot \frac{p_s}{p} + \frac{1}{T} \frac{d}{dx} \left( \frac{\kappa_v}{1 + \kappa_{pc}} \right) T_s \right\} \quad (4.17)$$

This has effectively separated terms of different order of magnitude by placing them on different sides of the equation. The Clausius-Clapeyron equation itself may be separated into variables of different orders, leaving it in the integrated form. Thus,

$$\frac{p}{p_c} - \exp \left\{ \frac{\gamma}{\gamma - 1} \eta \left( 1 - \frac{T_c}{T} \right) \right\} \\ \cong \frac{p_s}{p_c} - \frac{\gamma}{\gamma - 1} \eta \frac{T_c}{T} \frac{T_s}{T} \exp \left\{ \frac{\gamma}{\gamma - 1} \eta \left( 1 - \frac{T_c}{T} \right) \right\}. \quad (4.18)$$

The energy relation for the entire system, equation (4.6), may also be treated in a similar manner to obtain

$$\left( \frac{T}{T_c} - 1 \right) + \frac{u^2}{2c_p T_c} + \eta \frac{\kappa_{pc} - \kappa_p}{1 + \kappa_{pc}} \\ \cong \frac{\kappa_p}{1 + \kappa_{pc}} \left( \frac{T_s}{T_c} + \frac{u u_s}{c_p T_c} \right). \quad (4.19)$$

Under conditions of local equilibrium, the right-hand sides of Eqs. (4.17), (4.18), and (4.19) vanish and the left-hand sides correspond, with some rearrangement, to Eqs. (4.7), (4.8), and (4.9) for the "zeroth order" or equilibrium solution.

The so-called "slip quantities", defined in Eqs. (4.14), consist of only first order parts. Referring first to the equation of motion for the liquid phase, Eq. (4.4), it follows that  $u_s^{(1)}$  emerges directly as an algebraic function of known zeroth order quantities,

$$u_s^{(1)} = \frac{u^{(0)}}{a^{(0)}} \frac{du^{(0)}}{dx} \cdot \Sigma^{(0)^2} \quad (4.20)$$

and hence is known. The fact that no integration was required to obtain this result indicates that the differential order of the system has been suppressed, that is, the perturbation problem we have generated is a singular one. The physical consequence of this is that, although the steady slip of the droplet with respect to the vapor is described, the transition from an arbitrary initial state to the steady slip cannot be represented. As discussed in Section 2, this transient covers a distance of order  $\lambda_v$ . So long as  $\lambda_v$  is much smaller than the nozzle length  $L$ , this slight error is tolerated.

The variation  $p_s$  between the local pressure and the saturation pressure corresponding to the droplet temperature is obtained in a corresponding manner from Eqs. (4.1) together with (3.16). This yields

$$\frac{p_s^{(1)}}{p^{(0)}} = \frac{\lambda_R}{\lambda_v} \frac{u^{(0)}}{a^{(0)}} \frac{d}{dx} (\kappa_p^{(0)}) \cdot \Sigma^{(0)^2} \quad (4.21)$$

Finally, the difference  $T_s$  between vapor and droplet temperatures may be obtained from Eq. (4.5). Here, in addition to the process used in the two previous deductions, we must recognize that vapor production rate  $\mu_v \sim p_s$  and is therefore negligible in the present order of calculation. It follows directly then that

$$T_s^{(1)} = \frac{\lambda_T}{\lambda_v} \frac{u^{(0)}}{a^{(0)}} \frac{dT^{(0)}}{dx} \cdot \Sigma^{(0)^2} \quad (4.22)$$

The velocity, composition, and temperature gradients appearing in Eqs. (4.20), (4.21), and (4.22) may be written in terms of the prescribed pressure gradient through the zeroth order solution given previously. Thus, the zeroth order "slip" quantities are known completely. The quantity  $\Sigma^{(0)}$ , the equilibrium approximation to the droplet radius ratio, relates to the liquid mass flux. A little calculation shows that

$$\Sigma^{(0)} = \left( \frac{\dot{m}_p^{(0)}}{\dot{m}_{pc}} \right)^{\frac{1}{2}} = \left( \frac{\kappa_p^{(0)}}{\kappa_{pc}} \right)^{\frac{1}{2}} \quad (4.23)$$

and hence is known.

The first-order perturbations  $u^{(1)}$ ,  $T^{(1)}$ ,  $\kappa_v^{(1)}$ , etc., to the vapor properties require a little more calculation. If the expansions (4.14), (4.15) are substituted into the Clausius-Clapeyron equation, in the form given by (4.18), the left-hand side yields a zeroth-order term

identical with the equilibrium relation (4.7). After simplification, the first order part may be rearranged to give the vapor temperature perturbation

$$\frac{T^{(1)}}{T^{(0)}} = \frac{T_s^{(1)}}{T^{(0)}} - \frac{\gamma - 1}{\gamma} \frac{1}{\eta} \frac{T^{(0)}}{T_c} \frac{p_s^{(1)}}{p^{(0)}}. \quad (4.24)$$

The first-order perturbation terms on the right-hand side of Eq. (4.24) may alternatively be written in terms of the zeroth-order solution by means of Eqs. (4.21) and (4.22).

A similar type of process may be employed to obtain the first-order perturbation of vapor mass fraction  $\kappa_v^{(1)}$ . Substitution of the perturbation expansions into Eq. (4.17) gives a zeroth-order term on the left-hand side that is identical with the equilibrium relation (4.9). The first order parts, arising from the left-hand side, have terms involving derivatives of temperature and mass fraction corrections to the vapor phase; the right-hand side contains known first-order slip terms. Some rearrangement of these terms, several partial integrations, and use of the zeroth-order relations (4.7), (4.8), and (4.9) permit the differential equation for  $\kappa_v^{(1)}$  to be integrated from the chamber state,  $p_c$ , to the local point of interest described by the local pressure. Finally, use of the known value for  $T^{(1)}$ , Eq. (4.24), permits the integrated form to be expressed in terms of the first-order slip quantities and, in turn, by the equilibrium solution

$$\kappa_v^{(1)} = -(1 + \kappa_p^{(0)}) \frac{T_s^{(1)}}{T^{(0)}} + \frac{\gamma - 1}{\gamma} \frac{1}{\eta} \frac{T^{(0)}}{T_c} \frac{p_s^{(1)}}{p_c} + I(p) \quad (4.25)$$

where  $I(p)$  is the integral

$$I(p) = \frac{\gamma - 1}{\gamma} \int_{p_c}^p \left\{ \left[ -(\eta + \kappa_p^{(0)}) \right. \right. \\ \left. \left. \frac{1 + \kappa_{pc}}{\eta} \left( \log \frac{T_c}{T^{(0)}} - 1 \right) \right] \frac{T_s^{(1)}}{T_c} \right. \\ \left. + \frac{\gamma - 1}{\gamma} \left[ 1 - \frac{1 + \kappa_{pc}}{\eta} \left( \log \frac{T_c}{T^{(0)}} - 1 \right) \right] \left( \frac{T^{(0)}}{T_c} \right)^2 \frac{p_s^{(1)}}{p^{(0)}} \right. \\ \left. + \kappa_p^{(0)} \left[ \log \frac{T_c}{T^{(0)}} - \frac{\eta}{1 + \kappa_{pc}} \right] \frac{u_s^{(1)}}{u^{(0)}} \right\} \frac{1}{p} dp. \quad (4.26)$$

This is a simple quadrature inasmuch as the integrand involves known, well-defined functions of the local pressure  $p$ . There is no difficulty at the throat or minimum cross-sectional area of the nozzle because of the choice of integration variable.

Finally, the first-order modification to the vapor velocity may be obtained from the first order part of

Eq. (4.19). After some rearrangement and elimination of  $T^{(1)}$  and  $\kappa_v^{(1)}$  by means of Eqs. (4.24) and (4.25), the vapor velocity perturbation becomes

$$\frac{u^{(0)2}}{c_p T_c} \frac{u^{(1)}}{u^{(0)}} = - \left[ 1 + \frac{1 + \kappa_p^{(0)}}{1 + \kappa_{pc}} \eta \frac{T_c}{T^{(0)}} \right. \\ \left. - \frac{\kappa_p^{(0)}}{1 + \kappa_{pc}} \right] \frac{T_s^{(1)}}{T_c} - \frac{\gamma - 1}{\gamma} \frac{T^{(0)}}{T_c} \\ \times \left[ \frac{1}{1 + \kappa_{pc}} - \frac{1}{\eta} \left( \frac{T^{(0)}}{T_c} \right) \left( \frac{p_c}{p} \right) \frac{p_s^{(1)}}{p} \right] \\ + \frac{\kappa_p^{(0)}}{1 + \kappa_{pc}} \frac{u^{(0)2}}{c_p T_c} \frac{u_s^{(1)}}{u^{(0)}} - \frac{\eta}{1 + \kappa_{pc}} I(p) \quad (4.27)$$

The results of the first-order perturbation calculations which we have summarized in the preceding pages show how useful nozzle and channel flow problems may be solved with acceptable accuracy and a minimum of numerical computation. So long as the physical conditions of the problem are satisfied and the relaxation lengths  $\lambda_v$ ,  $\lambda_T$ ,  $\lambda_R$  are small in comparison to the distance along the flow direction over which large changes of state take place, this first-order solution is valid.

#### Normal shock in a wet vapor

The normal shock wave has been a favorite vehicle for the study of gas relaxation processes because it displays the details of the relaxation process so effectively along the flow direction. In particular, this has been utilized in two-phase flow or dusty gases by Carrier [12], Marble [10], Rudinger [13], and others. This technique is quite useful in the present system when the relaxation processes have such different time constants. Here, the structure of the shock may be divided into three distinct zones. As indicated in Fig. 2, the first of these is the normal gasdynamic shock which takes place in the vapor, without change in phase or state of the liquid droplets. This zone is thin, in the sense of conventional shock waves, having a thickness of a few mean free paths. In particular, it is thin in comparison with all other relaxation processes.

Now if we assume the two phases to be in equilibrium ahead of the shock, they are certainly not in equilibrium immediately behind the gasdynamic shock. The pressure of the vapor has increased to a value exceeding that corresponding to the droplet temperature. The condition of phase equilibrium is restored by the condensation of vapor upon each droplet surface with a relaxation time corresponding to  $\tau_R$ . This process continues until the enthalpy, including the latent heat, carried by the condensing vapor raises the droplet to the saturation temperature

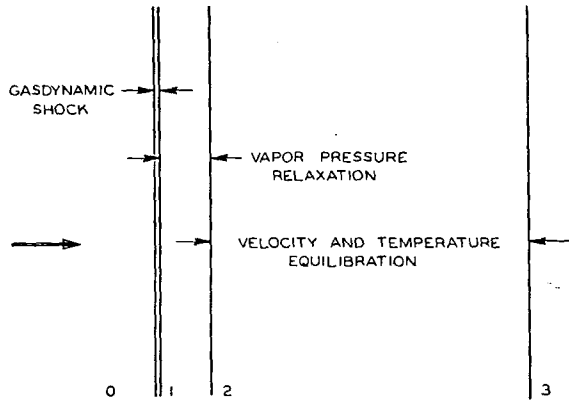


FIG. 2. Zones comprising the structure of normal shock in two-phase flow with condensation.

corresponding to the existing pressure of the vapor. Thus, although the liquid droplets will be much smaller or vanish downstream of the complete shock, their first change is to grow in response to the increase in pressure across the gasdynamic shock. The thickness of this vapor relaxation zone exceeds that of the gasdynamic shock by at least a factor of  $10^2$ , so long as the droplet radius  $\sigma$  is large in comparison with the mean free path.

Following the vapor relaxation zone, the temperature and velocity of the vapor still differ from those of the droplet. The final zone therefore consists of the relatively slow exchange of mass, momentum, and energy between the phases at a rate corresponding to the relaxation times  $\tau_T$  and  $\tau_v$ . The extent of this zone is at least  $10^2$  times as large as the vapor relaxation zone and may, under certain circumstances, be much larger. This region is the counterpart of the thermal and velocity equilibration zone discussed in Ref. 10 for inert particles.

The detailed structure of the two-phase shock structure, that is, the portion of it following the gasdynamic shock, may be computed utilizing the one-dimensional relations that we have given earlier in this section. This is, to a large extent, a computational task and rather than pursue it at this time we shall extract a considerable amount of information with physical significance by utilizing the fact that the relaxation processes are of differing orders of magnitude.

Because the gasdynamic shock, vapor relaxation, and velocity equilibration zones possess such different thicknesses, it is possible to calculate gross changes across each of them in the same general manner that one treats a conventional shock. For each zone, gross conservation relations may be written which include constraints appropriate to changes that are

permitted across each zone. In particular, the changes 0-1 across the gasdynamic shock are governed by the well-known conservation relations for a perfect gas, for the mechanical and thermal effects of the droplets upon the vapor may be disregarded in comparison with the internal viscous stresses within the vapor. Thus, we may assume the state known at station 1, the liquid state identical with that upstream, while the vapor state is related to the upstream condition through the conventional normal shock relation.

In the second region, the rapid phase transition controls the momentum and energy exchange between phases. The internal viscous stresses within the vapor and the viscous interaction of droplets and vapor contribute forces that are small in comparison with the momentum carried away by condensation. Then utilizing  $-\mu_v(x)$  to denote the local rate at which mass is being lost by the vapor phase, the equations of continuity, momentum, and energy are

$$\rho_2 u_2 = \rho_1 u_1 + \int_1^2 \mu_v dx = \rho_1 u_1 - \Delta \kappa_p \quad (4.28)$$

$$p_2 + \rho_2 u_2^2 = p_1 + \rho_1 u_1^2 + \int_1^2 \mu_v u dx \quad (4.29)$$

$$\rho_2 u_2 (h_2 + \frac{1}{2} u_2^2) = \rho_1 u_1 (h_1 + \frac{1}{2} u_1^2) + \int_1^2 \mu_v (h + \frac{1}{2} u^2) dx. \quad (4.30)$$

A similar and comparable set of relations describes the behavior of the droplet cloud. More conveniently these may be expressed as the relationships for the complete two-phase system

$$\rho_2 u_2 + \rho_{p2} u_{p2} = \rho_1 u_1 + \rho_{p0} u_{p0} \quad (4.31)$$

$$\rho_2 u_2^2 + \rho_{p2} u_{p2}^2 + p_2 = \rho_1 u_1^2 + \rho_{p0} u_{p0}^2 + p_1 \quad (4.32)$$

$$\rho_2 u_2 (h_2 + \frac{1}{2} u_2^2) + \rho_{p2} u_{p2} (e_{p2} + \frac{1}{2} u_{p2}^2) + \rho_1 u_1 (h_1 + \frac{1}{2} u_1^2) + \rho_{p0} u_{p0} (e_{p0} + \frac{1}{2} u_{p0}^2) \quad (4.33)$$

Now the end of the vapor relaxation zone is reached where the pressure  $p_2$  of the vapor and the temperature  $T_{p2}$  of the liquid droplets satisfy the thermodynamic saturation conditions. Utilizing the upstream equilibrium state as a reference, this is

$$\frac{p_2}{p_0} = \exp \left\{ \frac{h_l}{RT_0} \left( 1 - \frac{T_0}{T_{p2}} \right) \right\}. \quad (4.34)$$

It is not difficult to show that when  $h_l/RT_0$  is large, many of the changes between states 1 and 2 are small. For the most interesting range of alkali metal vapors, for example,  $5 \leq (h_l/RT) \leq 15$ , and therefore the condition is satisfied. As a consequence, the integrals in Eqs. (4.28), (4.29), and (4.30) may be approximated well by linear interpolation of the integrand between

the end points. For the same reason,  $|\Delta\kappa_p| \ll 1$  under these circumstances. Then some detailed calculation and approximation shows that the conditions at state 2 may, with good approximation, be expressed as

$$\Delta\kappa_p \cong \frac{\kappa_{p0} \log \frac{p_1}{p_0}}{\gamma\eta \left( \frac{\eta}{\gamma-1} + M_0^2 \right)} \quad (4.35)$$

$$\frac{p_2}{p_1} = 1 + \frac{\gamma M_1^2}{1 - M_1^2} \Delta\kappa_p \quad (4.36)$$

$$\frac{T_{p2}}{T_0} = \frac{1}{1 - \frac{\gamma-1}{\gamma} \frac{1}{\eta} \log \frac{p_1}{p_0}} \quad (4.37)$$

As one expects, the additional liquid mass flow  $\Delta\kappa_p$  is determined largely by the pressure ratio across the gasdynamic shock and the latent heat parameter  $\eta$ . It is clear that large values of  $\eta$  insure rather small mass exchange. The temperature rise of the droplets through this relaxation zone, however, is large and, although the particular expression given holds only when  $\Delta\kappa_p$  is small, its value is independent of  $\Delta\kappa_p$ .

The pressure rise, (4.36), is the familiar relation for the pressure rise in a subsonic flow with mass extraction. The results confirm our intuitive picture of vapor pressure and droplet temperature changing until the saturation condition is satisfied. Quantitatively, however, we see that the bulk of the accommodation takes place in the droplet temperature for large values of  $\eta$ . Recalling that the entire alteration of the droplet temperature rise originates with the sensible and latent heat carried by the condensing vapor, the large value of  $\eta$  assures that their change will be accomplished by a comparatively small mass exchange. As a consequence, the droplet growth is small.

The changes in state and composition across the final equilibration zone are described by a set of conservation relations comparable to those employed for the other two regions. The final state is determined by the attainment of complete equilibrium, that is, the velocities and temperatures of the two phases are equal and the two phases are in thermodynamic equilibrium. This zone is by far the largest of the three and constitutes the predominant portion of the shock structure.

Table 2. Variations of Vapor and Liquid State Across Each Zone of Shock Wave for Conditions  $\gamma = 1.6$ ,  $\kappa_{p0} = 0.25$

$M_0$	$h_l/RT_0$	1.5			2.0		
		5	10	15	5	10	15
$p_3/p_0$		3.52	4.0	4.19	6.35	7.30	7.48
	$p_1/p_0$	2.54	2.54	2.54	4.69	4.69	4.69
	$p_2/p_1$	1.02	1.01	1.00	1.02	1.01	1.00
	$p_3/p_2$	1.36	1.56	1.65	1.33	1.54	1.59
$T_3/T_0$		1.31	1.18	1.12	1.60	1.22	1.16
	$T_1/T_0$	1.46	1.46	1.46	1.98	1.98	1.98
	$T_2/T_1$	1.04	1.02	1.00	1.03	1.01	1.00
	$T_3/T_2$	0.86	0.79	0.77	0.79	0.61	0.58
$u_3/u_0$		0.44	0.33	0.33	0.34	0.21	0.19
	$u_1/u_0$	0.57	0.57	0.57	0.42	0.42	0.42
	$u_2/u_1$	0.97	0.99	1.00	0.97	0.99	1.00
	$u_3/u_2$	0.80	0.58	0.57	0.83	0.51	0.43
$T_{p3}/T_{p0}$		1.31	1.18	1.12	1.59	1.22	1.16
	$T_{p1}/T_{p0}$	1.00	1.00	1.00	1.00	1.00	1.00
	$T_{p2}/T_{p1}$	1.23	1.10	1.07	1.45	1.18	1.12
	$T_{p3}/T_{p2}$	1.06	1.08	1.05	1.10	1.03	1.05
$u_{p3}/u_{p0}$		0.44	0.33	0.33	0.34	0.21	0.19
	$u_{p1}/u_{p0}$	1.00	1.00	1.00	1.00	1.00	1.00
	$u_{p2}/u_{p1}$	0.94	0.98	0.99	0.93	0.97	0.99
	$u_{p3}/u_{p2}$	0.46	0.33	0.34	0.36	0.22	0.19
$\sigma_3/\sigma_0$		0.72	0.74	0.81	0.00	0.00	0.36
	$\sigma_1/\sigma_0$	1.00	1.00	1.00	1.00	1.00	1.00
	$\sigma_2/\sigma_1$	1.02	1.01	1.00	1.02	1.01	1.00
	$\sigma_3/\sigma_2$	0.71	0.74	0.81	0.00	0.00	0.36

The detailed changes across each of the three distinguishable zones are given in Table 2 for selected values of liquid mass fraction, initial Mach number, ratio of specific heats, and latent heat parameter  $h_l/RT_0$ . They are selected so as to indicate the effects of each of these parameters on the shock structure. The value of  $\kappa_{p0} = 0.25$  indicates a liquid mass fraction of 0.20 or a quality of 80 per cent. Results are given for two shock Mach numbers, 1.5 and 2.0, and for three values of the latent heat parameter,  $h_l/RT_0 = 5.0$ , 10.0, and 15.0. For each of these cases, the tabulation is arranged so that the ratio or change across the complete shock appears first and the changes across each of the structural zones appear below it. For example, when the parameters are  $M_0 = 2.0$ ,  $h_l/RT_0 = 5.0$ , the pressure ratio across the complete shock is 6.35. The pressure ratio across the vapor relaxation zone is 1.02, and the pressure ratio across the velocity and thermal equilibration zone is 1.33. It is to be noted that the dominant temperature rise for droplets occurs across the vapor relaxation zone for all examples, the value of the temperature rise becoming small for larger values of  $h_l/RT_0$  as indicated by Eq. (4.36).

Calculations of the structure of the velocity and thermal relaxation zone show that the thickness of this region, for the examples covered in the table, ranges between  $4\lambda_v$  and  $5\lambda_v$ .

### 5. Relaxation Processes and Propagation Speeds

Because they are so fundamental to problems of heterogeneous flow with phase change, the elementary relaxation processes of velocity, temperature, and composition were examined at some length in Section 2. In actual situations of physical interest, however, these elementary relaxation processes seldom occur in isolation. Rather, it is usual that these relaxation processes take place in some involved combination that makes their individual roles obscure. In the shock structure, for example, the composition relaxation mechanism plays a very different part in zone 2 and zone 3. It is very helpful, therefore, to examine some of these combined relaxation processes in circumstances so elementary that some general understanding may be drawn from their details.

#### *Temperature-composition relaxation at constant volume*

Let us consider a wet vapor at rest in an adiabatic container of constant volume. Under conditions of equilibrium, the state of the system is described by  $p_0$ ,  $T_0$ ,  $\rho_0$ ,  $\rho_{p0}$ , and the liquid phase is presumed uniformly dispersed in droplets of a single radius  $\sigma$ . If the system is displaced from equilibrium by a small

amount, consistent with the equilibrium state, we may ask the mode of decay of this small disturbance back to equilibrium. The process is described, referring to our general conservation relations developed in Section 3, by the general energy integral

$$\frac{d}{dt} \{ \rho_p c_p T_p + \rho c_p T + \rho h_l - p \} = 0 \quad (5.1)$$

the first law of thermodynamics for the liquid phase

$$c_p \frac{dT_p}{dt} = \frac{c_p(T - T_p)}{\tau_T} - \frac{h_l}{\tau_R} \left( \frac{p^e - p}{p} \right) \quad (5.2)$$

the vapor production rate

$$\frac{d\rho}{dt} = \frac{\rho_p}{\tau_R} \left( \frac{p^e - p}{p} \right) \quad (5.3)$$

the conservation of mass

$$\rho + \rho_p = \text{constant} \quad (5.4)$$

the Clausius-Clapeyron equation

$$\frac{p^e}{p_0} = \exp \left\{ \frac{h_l}{RT_0} \left( 1 - \frac{T_0}{T_p} \right) \right\} \quad (5.5)$$

where the equilibrium state is used as a reference. In discussing the perturbation problem, it is convenient to introduce

$$\left. \begin{aligned} p' &= \frac{p - p_0}{p_0} & p^{e'} &= \frac{p^e - p_0}{p_0} \\ T' &= \frac{T - T_0}{T_0} & T'_p &= \frac{T_p - T_0}{T_0} \\ \rho' &= \frac{\rho - \rho_0}{\rho_0} & \rho'_p &= \frac{\rho_p - \rho_{p0}}{\rho_{p0}} \end{aligned} \right\} \quad (5.6)$$

Then the linearized form of (5.1), integrated under the conditions that the reference equilibrium state be maintained, is

$$\kappa_{p0} T'_p + T' + \eta \rho' - \frac{\gamma - 1}{\gamma} p' = 0 \quad (5.7)$$

where  $\kappa_{p0} = \rho_{p0}/\rho_0$ . In a corresponding manner, Eq. (5.2), (5.3), (5.4), and the Clausius-Clapeyron relation becomes respectively

$$\frac{dT'_p}{dt} = \frac{1}{\tau_T} (T' - T'_p) - \frac{\eta}{\tau_R} (p^{e'} - p') \quad (5.8)$$

$$\frac{d\rho'}{dt} = \frac{\kappa_{p0}}{\tau_R} (p^{e'} - p') \quad (5.9)$$

$$\rho' + \rho'_p = 0 \quad (5.10)$$

$$p^{e'} = \frac{\gamma}{\gamma - 1} \eta T'_p \quad (5.11)$$

and finally the equation of state for the vapor is

$$p' = \rho' + T'. \quad (5.12)$$

It is a straightforward operation to eliminate  $p'$ ,  $\rho'$ , and  $\rho'_p$  among these and obtain two first-order differential and one algebraic equations in  $T'$ ,  $p'$ , and  $T'_p$

$$\frac{d}{dt}(T') - \left(\frac{d}{dt} + \frac{\kappa_{p0}}{\tau_R}\right)p' + \frac{\kappa_{p0}}{\tau_R} \frac{\gamma}{\gamma-1} \eta T'_p = 0 \quad (5.13)$$

$$\frac{1}{\tau_T} T' + \frac{\eta}{\tau_R} p' - \left(\frac{d}{dt} + \frac{1}{\tau_T} + \frac{\gamma}{\gamma-1} \frac{\eta^2}{\tau_R}\right) T'_p = 0 \quad (5.14)$$

$$(\eta-1)T' - \left(\eta - \frac{\gamma-1}{\gamma}\right)p' - \kappa_{p0}T'_p = 0. \quad (5.15)$$

For this linear system with constant coefficients, choose solutions  $T^* \exp(-t/\tau)$ ,  $p^* \exp(-t/\tau)$ ,  $T'_p \exp(-t/\tau)$ ;  $T^*$ ,  $p^*$ ,  $T'_p$  being the initial perturbation magnitudes and  $\tau$  the characteristic decay time. The characteristic decay time constants are obtained by setting to zero the determinant of the coefficient of the set obtained by substituting the assumed solutions.

$$\begin{vmatrix} -\frac{1}{\tau} & -\left(-\frac{1}{\tau} + \frac{\kappa_{p0}}{\tau_R}\right) & \frac{\kappa_{p0}}{\tau_R} \frac{\gamma}{\gamma-1} \eta \\ \frac{1}{\tau_R} & \frac{\eta}{\tau_R} & -\left(\frac{1}{\tau} + \frac{1}{\tau_T} + \frac{\gamma}{\gamma-1} \frac{\eta^2}{\tau_R}\right) \\ \eta-1 & -\left(\eta - \frac{\gamma-1}{\gamma}\right) & -\kappa_{p0} \end{vmatrix} = 0 \quad (5.16)$$

Since this is of the second algebraic order in  $1/\tau$ , there are two different possible decay rates. Both roots are real and negative, but for usual values of the parameters they are of quite different sizes. The exact values of the roots are somewhat too involved to show this clearly, but it is easily demonstrated when the value of  $\kappa_{p0}$  is small and  $\eta$ , the latent heat-parameter, is large. Then these roots are

$$\frac{1}{\tau_1} \cong \gamma \kappa_{p0} \left( \frac{1 - 2\eta + \frac{\gamma}{\gamma-1} \eta^2}{\tau_R + \tau_T \frac{\gamma}{\gamma-1} \eta^2} \right) \quad (5.17)$$

$$\frac{1}{\tau_2} \cong \left( \frac{1}{\tau_T} + \frac{1}{\tau_R} \frac{\gamma}{\gamma-1} \eta^2 \right) \quad (5.18)$$

where clearly  $\tau_1 \gg \tau_2$  since it is proportional to the large quantity  $1/\kappa_{p0}$ .

These two decay constants, in the same manner as characteristic frequencies of an oscillating system,

imply "natural modes" in the sense that a choice of initial conditions may be made so that the decay will take place with only one of the time constants. For our problem, this implies that a certain choice of  $T^*$ ,  $p^*$ , and  $T'_p$  will excite only  $\tau_1$ , the long decay time solution. This is easily determined to be

$$T^*(\tau_1) = p^*(\tau_1) \quad (5.19)$$

$$T'_p(\tau_1) \sim \frac{\gamma-1}{\eta} p^*(\tau_1). \quad (5.20)$$

In this initial state, the vapor pressure and temperature and the droplet temperature have been increased above equilibrium by vaporizing material. The mass to be condensed is proportional to the vapor mass, and the droplet area through which the transfer must be effected is small, because of small liquid mass. Thus, the time required is long.

The short decay time, on the other hand, leads to

$$p^*(\tau_2) = -\frac{\kappa_{p0}}{\eta^2} T'_p(\tau_2) \quad (5.21)$$

$$T^*(\tau_2) = \frac{\kappa_{p0}}{\eta} T'_p(\tau_2). \quad (5.22)$$

For this initial state, the vapor pressure is slightly above and the vapor temperature slightly below equilibrium; the droplet temperature is below equilibrium by an order of magnitude unity. The relaxation process requires very small change in vapor state and mass transfer from vapor to droplet. However, this small condensation on the droplets is sufficient to raise their temperature to equilibrium. Because of the small droplet mass, the process is rapid because of the relatively small mass to be exchanged.

For a general relaxation process, the course of events divides in two parts. First, the droplets will grow at the rapid rate given by  $\tau_2$  until all deviations from equilibrium are positive and of the same order of magnitude. Then the decay will continue at the very slow rate given by the relaxation constant  $\tau_1$ . It is this division we observe in regions 1-2 and 2-3 in the shock wave structure.



### Sound propagation with constraints

Another effective manner of examining relaxation processes involving combinations of the momentum, heat-, and mass-transfer mechanisms is by examination of the sound propagation speeds. By the process of choosing some of the time constants very long in comparison with the time for passage of the sound wave, the quantity associated with that time constant is effectively unchanged during the process, that is, it is "frozen." In this manner, it is possible to extract propagation speeds or time constants associated with the response of the system constrained in various manners. These quantities will in fact appear as physical parameters in subsequent considerations.

Consider a one-dimensional flow of the two-phase mixture in which property changes  $du$ ,  $d\rho$ ,  $dp$ , ... take place across a stationary gradient of length  $dx$ . The one-dimensional small disturbance relations corresponding to Eqs. (3.1), (3.5), (3.6), (3.11), (3.13), the equation of state and (3.18) are

$$\frac{du}{u_0} + \frac{d\rho}{\rho_0} = \kappa_{p0} \frac{a_0}{u_0} \left( \frac{p^e - p}{p_0} \right) \frac{dx}{\lambda_R} \quad (5.23)$$

$$\frac{\rho_0 u_0^2}{p_0} \frac{du}{u_0} + \frac{dp}{p_0} = \gamma \kappa_{p0} \left( \frac{u_p - u}{a_0} \right) \frac{dx}{\lambda_v} \quad (5.24)$$

$$\frac{\rho_0 u_0^2}{p_0} \frac{du_p}{u_0} = -\gamma \left( \frac{u_p - u}{a_0} \right) \frac{dx}{\lambda_v} \quad (5.25)$$

$$\frac{dT}{T_0} - \frac{\gamma - 1}{\gamma} \frac{dp}{p_0} = \kappa_{p0} \frac{a_0}{u_0} \left( \frac{T_p - T}{T_0} \right) \frac{dx}{\lambda_T} \quad (5.26)$$

$$\begin{aligned} \frac{dT_p}{T_0} = & -\frac{a_0 c_p}{u_0 c} \left( \frac{T_p - T}{T_0} \right) \frac{dx}{\lambda_T} \\ & - \frac{a_0}{u_0} \eta \frac{c_p}{c} \left( \frac{p^e - p}{p_0} \right) \frac{dx}{\lambda_R} \end{aligned} \quad (5.27)$$

$$\frac{dp}{p_0} - \frac{d\rho}{\rho_0} - \frac{dT}{T_0} = 0 \quad (5.28)$$

$$\frac{dp^e}{p_0} - \frac{\gamma}{\gamma - 1} \eta \frac{dT_p}{T_0} = 0. \quad (5.29)$$

Each of the relaxation processes appears on the right-hand sides of these equations and contains two essential factors, (i) the ratio of the distance along the flow path  $dx$  to the relaxation length, and (ii) the difference between the two variables that are relaxing toward each other. For example, the equation of motion for droplets (5.25) contains a right-hand side with one factor  $dx/\lambda_v$  and one factor  $(u_p - u)/u_0$ , the ratio of the particle slip velocity to the undisturbed velocity.

Now consider  $dx$  to be fixed, a measure of the wave thickness. If some of the relaxation processes are "very rapid", we imply that their relaxation lengths are very small in comparison with the wave thickness. For example, if the velocity equilibration and the vapor pressure relaxation are "very rapid",  $dx/\lambda_v \gg 1$ ,  $dx/\lambda_R \gg 1$ . But referring to Eq. (5.25), clearly the velocity difference  $u_p - u \rightarrow 0$  so that the product  $-\gamma[(u_p - u)/u_0](dx/\lambda_v)$  remains finite. Therefore, it appears that the limit of Eq. (5.25), as the velocity equilibration length becomes very small,

$$\frac{\rho_0 u_0^2}{p_0} \frac{du_p}{u_0} \rightarrow \frac{\rho_0 u_0^2}{p_0} \frac{du}{u_0} = \lim_{dx/\lambda_v \rightarrow \infty} \left\{ -\gamma \frac{u_p - u}{u_0} \frac{dx}{\lambda_v} \right\}.$$

Substituting this result for the right-hand side of Eq. (5.24), we find that its limit as  $dx/\lambda_v \rightarrow 0$  becomes

$$(1 + \kappa_{p0}) \frac{\rho_0 u_0^2}{p_0} \frac{du}{u_0} + \frac{dp}{p_0} = 0. \quad (5.30)$$

The limiting process leads to the elimination of one unknown,  $u_p$ , and one equation, (5.25). Because of its relative rapidity, the velocity relaxation process does not appear in our problem; we refer to it as being in equilibrium, implying that for the processes we are investigating, the slip velocity is relatively negligible, but the forces which are required to move the droplet in this manner must be accounted for.

Now when we further specify that the vapor pressure relaxation is very fast, that is, that  $dx/\lambda_R \rightarrow \infty$ , we determine the limiting value of  $a_0/u_0 [(p^e - p)/p_0]$  from Eq. (5.23) and substitute in the particle temperature equation to give

$$\begin{aligned} \frac{dT_p}{T_0} + \frac{(c_p/c)\eta}{\kappa_{p0}} \left( \frac{du}{u_0} + \frac{d\rho}{\rho_0} \right) \\ = -\frac{a_0 c_p}{u_0 c} \left( \frac{T_p - T}{T_0} \right) \frac{dx}{\lambda_T} \end{aligned} \quad (5.31)$$

and because  $p^e - p \rightarrow 0$ , Eq. (5.29) becomes

$$\frac{dp}{p_0} - \frac{\gamma}{\gamma - 1} \eta \frac{dT_p}{T_0} = 0. \quad (5.32)$$

The single relaxation process remaining is the thermal relaxation process, measured by the thermal relaxation length  $\lambda_T$ . Assume that the thickness of the wave we are considering is small in comparison. Then for a finite temperature difference  $T_p - T$ ,  $dx/\lambda_T \rightarrow 0$  as  $dx \rightarrow 0$ ,  $dx/\lambda_R$ ,  $dx/\lambda_v \ll 1$ . The set of relations describing the wave, Eq. (5.23)–(5.29), is thus reduced to homogeneous set:

$$\begin{bmatrix} \frac{\gamma u_0^2}{a^2} & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{\gamma-1}{\gamma} & 1 & 0 \\ \frac{\eta}{\kappa_{p0}} \cdot \frac{c_p}{c} & \frac{\eta}{\kappa_{p0}} \cdot \frac{c_p}{c} & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -\frac{\gamma}{\gamma-1} \eta \end{bmatrix} \begin{Bmatrix} \frac{du}{u_0} \\ \frac{d\rho}{\rho_0} \\ \frac{dp}{p_0} \\ \frac{dT}{T_0} \\ \frac{dT_p}{T_0} \end{Bmatrix} = 0. \quad (5.33)$$

This set has a non-trivial solution, that is, the wave we describe exists, if the determinant of the coefficient vanishes, that is, if its propagation speed  $u_0$  satisfies the relation

$$u_0^2 (\equiv a_{vR}^2) = \frac{a^2}{(1 + \kappa_{p0}) \left( 1 + \frac{(\gamma-1) \kappa_{p0} (c/c_p)}{\eta^2} \right)} \quad (5.34)$$

where  $a = \sqrt{(\gamma p_0/\rho_0)}$  is the isentropic sound speed in the vapor alone. Physically, this wave speed is that which would occur if (i) the droplets and the vapor moved together without slip, (ii) the phase change process was so rapid that the vapor pressure was everywhere equal to the saturation pressure corresponding to the droplet temperature, and (iii) the droplet temperature and the vapor temperature were uncoupled. We speak then of the velocity and vapor pressure relaxation processes being "equilibrated" and the droplet temperature relaxation as "frozen", and indicate the corresponding wave propagation speed as  $a_{vR}$ , denoting by subscripts the equilibrated processes. Clearly, there are a number of such speeds that may be calculated; the results are given in Table 3. In the table, the notation under "Relaxation Processes" is such that  $\infty$  signifies that the relaxation process is "frozen"; 0 signifies that it is in equilibrium.

## 6. Wave Motion and the Effects of Exchange Processes

The manner in which the various characteristic times or relaxation lengths enter into a gasdynamic problem with phase change is exhibited quite conveniently by considering steady or moving weak waves in a uniform rectilinear flow. Consider a two-phase flow of a pure substance in which the vapor and droplets have a uniform velocity parallel to the

$x$ -axis in the undisturbed state. The vapor and droplets are in thermodynamic equilibrium with a common temperature  $T_0$  and a saturation pressure  $p^e(T_0) \equiv p_0$ . In this unperturbed state, the density of the vapor and the density of the liquid droplets are  $\rho_0$  and  $\rho_{p0}$ , respectively, and have a ratio  $\rho_{p0}/\rho_0 = \kappa$ , which we use to define the concentration level of the liquid phase.

The linearization of the governing equations proceeds in the conventional manner. For example, the continuity equation for the vapor, Eq. (3.1), by noting that

$$\frac{|\rho - \delta_0|}{\rho_0} \ll 1, \quad \frac{|u_1 - u_0|}{u_0} \ll 1, \quad \frac{|u_2|}{u_0} \ll 1$$

and in the relation for  $\mu_v$ , Eq. (3.6), that both  $p^e(T_p)$  and  $p$  differ only slightly from  $p_0$ , may be written in the form

$$\frac{\partial}{\partial x_j} \left( \frac{u_j}{u_0} \right) + \frac{\partial}{\partial x_1} \left( \frac{\rho}{\rho_0} \right) + \frac{\kappa}{M_0 \lambda_R} \left( \frac{p}{p_0} - \frac{p^e}{p_0} \right) = 0. \quad (6.1)$$

Here, we denote by  $u_j$ ,  $p$ ,  $\rho$  the perturbation of the physical quantities from their equilibrium values, so that actually  $u_1$  denotes  $u_1 - u_0$ ,  $u_2$  denotes  $u_2$ , etc. Similarly,  $p^e$  denotes  $p^e(T_p) - p^e(T_0)$ , the variation of the equilibrium vapor pressure from its undisturbed value. The Mach number  $M_0$  represents the ratio of the undisturbed gas velocity parallel to the  $x$ -axis to the undisturbed sound velocity in the vapor considered as a perfect gas.

Similarly, the continuity equation for the liquid phase, (3.2), becomes

$$\kappa \frac{\partial}{\partial x_j} \left( \frac{u_{pj}}{u_0} \right) + \frac{\partial}{\partial x_1} \left( \frac{\rho_p}{\rho_0} \right) - \frac{\kappa}{M_0 \lambda_R} \left( \frac{p}{p_0} - \frac{p^e}{p_0} \right). \quad (6.2)$$

In both these equations, it is to be noted that the

Table 3

Square of wave speed	Value	Relaxation processes		
		$\lambda_v$	$\lambda_T$	$\lambda_R$
$a^2$	$\gamma p_0/\rho_0$	$\infty$	$\infty$	$\infty$
$a_v^2$	$\frac{1}{1+\kappa_p} a^2$	0	$\infty$	$\infty$
$a_T^2$	$\frac{c_p + c\kappa_p}{c_p + \gamma c\kappa_p} a^2$	$\infty$	0	$\infty$
$a_R^2$	$\frac{1}{1 + \frac{(\gamma-1)\kappa_p(c/c_p)}{\eta^2}} a^2$	$\infty$	$\infty$	0
$a_{vT}^2$	$\frac{c_p + c\kappa_p}{(1+\kappa_p)(c_p + \gamma c\kappa_p)} a^2$	0	0	$\infty$
$a_{vR}^2$	$\frac{1}{(1+\kappa_p) \left[ 1 + \frac{(\gamma-1)\kappa_p(c/c_p)}{\eta^2} \right]} a^2$	0	$\infty$	0
$a_{TR}^2$	$\frac{1}{\gamma - \frac{2(\gamma-1)}{\eta} + \frac{(\gamma-1)[1+\kappa_p(c/c_p)]}{\eta^2}} a^2$	$\infty$	0	0
$a_{vTR}^2$	$\frac{1}{(1+\kappa_p) \left[ \gamma - \frac{2(\gamma-1)}{\eta} + \frac{(\gamma-1)[1+\kappa_p(c/c_p)]}{\eta^2} \right]} a^2$	0	0	0

variation of droplet radius does not enter, since it would multiply terms that are already perturbation quantities.

Linearization of the remaining relations proceeds in a similar manner. The equations of motion for the vapor phase, omitting the viscous stresses, are

$$\gamma M_0^2 \frac{\partial}{\partial x_1} \left( \frac{u_i}{u_0} \right) + \gamma M_0 \frac{\kappa}{\lambda_v} \left( \frac{u_i}{u_0} \right) - \gamma M_0 \frac{\kappa}{\lambda_v} \left( \frac{u_{pi}}{u_0} \right) + \frac{\partial}{\partial x_1} \left( \frac{p}{p_0} \right) = 0 \quad (6.3)$$

while the corresponding equations for the liquid phase are

$$M_0 \frac{\partial}{\partial x_1} \left( \frac{u_{pi}}{u_0} \right) + \frac{1}{\lambda_v} \frac{u_{pi}}{u_0} - \frac{1}{\lambda_v} \frac{u_i}{u_0} = 0. \quad (6.4)$$

The energy equation for the vapor phase, (3.11), simplifies because we omit the terms arising from the viscous dissipation and the conduction of heat, and because the dissipation associated with droplets moving through the vapor is a second-order term  $\sim (u_{pi} - u_i)^2$ . For the same reason, the term  $\mu_v [e(T_p) - e(T)]$  is negligible. Then writing the variation of vapor enthalpy  $\delta h$  as  $c_p \delta T$ , the linearized form is

$$M_0 \frac{\partial}{\partial x_1} \left( \frac{T}{T_0} \right) + \frac{\kappa}{\lambda_T} \left( \frac{T}{T_0} \right) - \frac{\kappa}{\lambda_T} \left( \frac{T_p}{T_0} \right) - \frac{\gamma-1}{\gamma} M_0 \frac{\partial}{\partial x_1} \left( \frac{p}{p_0} \right) = 0. \quad (6.5)$$

The energy equation for the droplets, (3.13), is linearized in a straightforward manner after taking  $\delta e_p = c \delta T_p$  and replacing the vapor production term  $\mu_v$  by its representation from (2.24). This results in

$$\kappa \frac{c}{c_p} M_0 \frac{\partial}{\partial x_1} \left( \frac{T_p}{T_0} \right) + \frac{\kappa}{\lambda_T} \left( \frac{T_p}{T_0} \right) - \frac{\kappa}{\lambda_T} \left( \frac{T}{T_0} \right) - \frac{\kappa \eta}{\lambda_R} \left( \frac{p}{p_0} - \frac{p^e}{p_0} \right) = 0 \quad (6.6)$$

where, by  $\eta$ , we mean

$$\eta = \frac{h_l}{c_p T_0}. \quad (6.7)$$

Through direct linearization, the equation of state for the vapor becomes

$$\frac{p}{p_0} = \frac{\rho}{\rho_0} + \frac{T}{T_0} \quad (6.8)$$

and the Clausius-Clapeyron relation gives  $p^e$ , the variation of the equilibrium vapor pressure  $p^e$

corresponding to the local particle temperature from the equilibrium vapor pressure  $p_0$  at the reference temperature  $T_0$ ,

$$\frac{p^e}{p_0} = \frac{\gamma}{\gamma - 1} \eta \frac{T_p}{T_0}. \quad (6.9)$$

Finally, the equation for droplet radius ratio, (3.17), is linearized to

$$M_0 \frac{\partial \Sigma}{\partial x_1} = \frac{1}{3\lambda_R} \left( \frac{p - p^e}{p_0} \right). \quad (6.10)$$

Two features of these linearized equations are notable in simplifying our further calculations. First, Eqs. (6.2) and (6.10) are decoupled from the remainder because the unknown  $\rho_p$  appears only in (6.2) and  $\Sigma$  only in (6.10). Therefore, the remaining equations may be solved without reference to (6.2) and (6.10) and subsequently they may be used to compute  $\rho_p$  and  $\Sigma$  from the other known quantities.

Secondly, a flow field obtained by small disturbance from an equilibrium irrotational flow field remains irrotational in *both* the vapor and droplet phases. This conclusion follows directly by taking the curl of Eq. (6.3) and of Eq. (6.4) and computing the change of vorticity in either phase as the media are convected parallel to the  $x_1$  axis. That these vorticity components retain their initial zero values is a property of only the linearized equations and not of the non-linear equations with transport properties suppressed.

As a consequence of these results, it is appropriate to work with Eqs. (6.1), (6.3), (6.4), (6.5), (6.6), (6.8), and (6.9), introducing also the perturbation velocity potentials  $\varphi$  and  $\varphi_p$  such that

$$\begin{aligned} \frac{u_i}{u_0} &= \frac{\partial \varphi}{\partial x_i} \\ \frac{u_{pi}}{u_0} &= \frac{\partial \varphi_p}{\partial x_i}. \end{aligned}$$

Choose the two potentials  $\varphi, \varphi_p$ ; the two temperature ratios,  $T/T_0, T_p/T_0$ , and the difference between local vapor pressure and droplet saturation pressure  $(p - p^e)/p_0$  as the dependent variables. Utilizing the equation of state, (6.8), and the Clausius-Clapeyron relation (6.9), Eq. (6.1) becomes

$$\begin{aligned} \nabla^2 \varphi - \frac{\partial}{\partial x_1} \left( \frac{T}{T_0} \right) + \frac{\gamma}{\gamma - 1} \eta \frac{\partial}{\partial x_1} \left( \frac{T_p}{T_0} \right) \\ + \frac{1}{M_0} \left( M_0 \frac{\partial}{\partial x_1} + \frac{\kappa}{\lambda_R} \right) \left( \frac{p - p^e}{p_0} \right) = 0 \end{aligned} \quad (6.11)$$

where the potential  $\varphi$  for gas velocity perturbation has also been introduced. Substitution of the potentials permits integration of Eq. (6.4) into a single relation

$$\left( M_0 \frac{\partial}{\partial x_1} + \frac{1}{\lambda_v} \right) \varphi_p - \frac{1}{\lambda_v} \varphi = 0. \quad (6.12)$$

Equations (6.3), equations of motion for the vapor phase, may also be integrated after substituting the potentials

$$\begin{aligned} M_0 \left( M_0 \frac{\partial}{\partial x_1} + \frac{\kappa}{\lambda_v} \right) \varphi - M_0 \frac{\kappa}{\lambda_v} \varphi_p \\ + \frac{\eta}{\gamma - 1} \left( \frac{T_p}{T} \right) + \frac{1}{\gamma} \left( \frac{p - p^e}{p_0} \right) = 0. \end{aligned} \quad (6.13)$$

The Clausius-Clapeyron equation has also been used to compute the term  $(1/\gamma)[(p - p^e)/p_0]$ . The same device is used to write the vapor energy equation in the form

$$\begin{aligned} \left( M_0 \frac{\partial}{\partial x_1} + \frac{\kappa}{\lambda_T} \right) \left( \frac{T}{T_0} \right) - \left( \eta M_0 \frac{\partial}{\partial x_1} + \frac{\kappa}{\lambda_T} \right) \left( \frac{T_p}{T_0} \right) \\ - \frac{\gamma - 1}{\gamma} M_0 \frac{\partial}{\partial x_1} \left( \frac{p - p^e}{p_0} \right) = 0 \end{aligned} \quad (6.14)$$

Finally, the energy equation for the liquid phase follows from Eq. (6.6).

Now the determinant of the coefficients of this linear homogeneous set of equations (6.11), (6.12), (6.13), (6.14), (6.6), represents essentially the differential equation for any one of the unknown functions. To make our analysis definite, consider the differential equation for the perturbation potential of the vapor flow,  $\varphi$ . The resulting equation is of the fifth order and may be written in the form

$$\begin{aligned} \left( M_0 \frac{\partial}{\partial x_1} \right)^3 \square^2 \varphi + \left( M_0 \frac{\partial}{\partial x_1} \right)^2 \\ \times \left\{ \frac{1}{A_v} \square_v^2 \varphi + \frac{1}{A_T} \square_T^2 \varphi + \frac{1}{A_R} \square_R^2 \varphi \right\} \\ + \left( M_0 \frac{\partial}{\partial x_1} \right) \left\{ \frac{1}{A_v} \frac{1}{A_T} \square_v^2 \varphi + \frac{1}{A_v} \frac{1}{A_R} \square_v^2 \varphi \right. \\ \left. + \frac{1}{A_T} \frac{1}{A_R} \square_T^2 \varphi \right\} \\ + \left( \frac{\kappa(c/c_p)}{1 + \kappa(c/c_p)} \right) \frac{1}{A_v} \frac{1}{A_T} \frac{1}{A_R} \square_v^2 \varphi = 0. \end{aligned} \quad (6.15)$$

Here, the quantities  $A_v, A_T, A_R$  are the modified relaxation lengths

$$\left. \begin{aligned} A_v &= \lambda_v \\ A_T &= \frac{1}{(c_p/c) + \kappa} \lambda_T \\ A_R &= \frac{1}{\frac{\gamma}{\gamma - 1} \frac{c_p}{c} \eta^2} \lambda_R \end{aligned} \right\} \quad (6.16)$$

and the  $\square$ 's are wave operators; for example,

$$\square_{vR}^2 \equiv \left( \frac{u_0}{a_{vR}^2} - 1 \right) \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}. \quad (6.17)$$

The sound speed  $a_{vR}$  and comparable ones entering other relations, are the speeds with which a weak wave propagates through the two-phase mixture with the constraint that the velocities and temperatures of the two phases are locally equal. They are identical, in fact, with those constrained sound speeds found in section 5 by one-dimensional considerations. They are given completely in Table 3.

It is particularly to be noted that this form, and the interpretation of the constrained propagation speeds, corresponds to the extensive analysis of gasdynamics of chemically reactive mixtures carried out by Napolitano [14]. Many of his results and observations may be carried over directly to the multiphase flow problem.

Each of the wave operators, such as  $\square_{vR}^2$ , involves a Mach number based upon the free stream velocity  $u_0$  and the constrained sound speed indicated by the subscripts on the wave operator. The general wave

$$\left\{ \frac{1}{a_2^2} = \frac{1}{a^2} \frac{\frac{1}{A_v A_T a_{vT}^2} + \frac{1}{A_v A_R a_{vR}^2} + \frac{\kappa(c/c_p)}{[1 + \kappa(c/c_p)] A_T A_R a_{TR}^2}}{\frac{1}{A_v A_T} + \frac{1}{A_v A_R} + \frac{\kappa(c/c_p)}{[1 + \kappa(c/c_p)] A_T A_R}} \right\} \quad (6.19)$$

equation (6.15) then involves wave operators based upon each of the eight constrained sound speeds listed in Table 3. Although this form is not particularly convenient for obtaining solutions, it does provide some physical insight into the mathematical formulation of the problem. The effects of "freezing" or "equilibrating" certain of the physical processes is directly shown; for in "freezing", equilibration is never achieved, and the characteristic length associated with that process becomes infinitely large. For "equilibrium" processes, the relaxation processes are infinitely fast, so that the characteristic length associated with an equilibrium process becomes zero. To illustrate this, suppose the temperature and vapor pressure relaxation processes are frozen, that is,  $A_R, A_T \rightarrow \infty$ . Then

$$M_0 \frac{\partial}{\partial x_1} \square^2 \varphi + \frac{1}{A_v} \square_v^2 \varphi = 0$$

which describes the two-phase flow of solid, thermally isolated particles with a gas. For all relaxation processes frozen,

$$\square^2 \varphi = 0$$

corresponding to wave propagation in the vapor only. For the velocity and temperature in equilibrium, however,  $A_v, A_T \rightarrow 0$  we have

$$\left( M_0 \frac{\partial}{\partial x_1} \right) \square_{vT}^2 \varphi + \frac{\kappa(c/c_p)}{1 + \kappa(c/c_p)} \frac{1}{A_R} \square_{vTR}^2 \varphi = 0$$

which describes the flow of very small droplets with the vapor in which the vapor pressure is out of equilibrium. For all relaxation processes in equilibrium,

$$\square_{vTR}^2 = 0.$$

These conclusions confirm our physical intuition that the wave propagation problem is governed by the constrained sound speed that enters into the highest order differential operator in each case.

For analytical purposes, it is convenient to define two new sound speeds

$$\frac{1}{a_1^2} = \frac{1}{a^2} \left\{ \frac{\frac{1}{A_v a_v^2} + \frac{1}{A_T a_T^2} + \frac{1}{A_R a_R^2}}{\frac{1}{A_v} + \frac{1}{A_T} + \frac{1}{A_R}} \right\} \quad (6.18)$$

These are in each case composites of those we have examined previously but do not have the direct physical interpretation of the components. The two corresponding wave operators  $\square_1^2$  and  $\square_2^2$ , constructed in the same spirit as the example given in (6.17), permit writing the general wave equation in the form

$$\begin{aligned} & \left( M_0 \frac{\partial}{\partial x_1} \right)^3 \square^2 \varphi + \left( \frac{1}{A_v} + \frac{1}{A_T} + \frac{1}{A_R} \right) \\ & \left( M_0 \frac{\partial}{\partial x_1} \right)^2 \square_1^2 \varphi + \left( \frac{1}{A_v A_T} + \frac{1}{A_v A_R} \right. \\ & \left. + \frac{\kappa(c/c_p)}{[1 + \kappa(c/c_p)] A_T A_R} \right) \left( M_0 \frac{\partial}{\partial x_1} \right) \square_2^2 \varphi \\ & \left. + \frac{\kappa(c/c_p)}{[1 + \kappa(c/c_p)] A_v A_T A_R} \square_{vTR}^2 \varphi. \right\} \quad (6.20) \end{aligned}$$

This is now of a recognizable type, particular cases of which have been discussed by Ackeret [15], Moore [16], B. T. Chu [17], J. T. C. Liu [18], L. Zung [19]

and others. The general wave propagation problem discussed by equations of this mathematical form was discussed in detail by Whitham [20], and it is in terms of his results that we can give a physical interpretation to solutions of (6.20).

The initial portion of the waves or "signal" is propagated with the sound speed of the vapor alone, the frozen sound speed. The strength of this fast wave is damped exponentially in distance from the point of its initiation. At sufficiently large distances from the point of initiation, the bulk of the wave propagates at the completely equilibrium sound speed  $a_{vTR}$ . Intermediate waves travel at speeds  $a_1$  and  $a_2$ ; each of these is damped, the damping becoming more gradual for lower order of the derivatives appearing in the operator. In the present physical case,

$$a > a_1 > a_2 > a_{vTR}$$

which insures stability of the simple wave. The details of wave structure and the rates of damping for the three damped waves require analysis of a particular problem and will not be attempted here.

### 7. Effects of Phase Exchange on Attenuation and Dispersion of Sound

Another question, that arises naturally from the wave propagation phenomena discussed in the previous section, is the nature of sound propagation through a medium consisting of a gas containing liquid droplets that may undergo phase exchange during the sound propagation process. Sound propagation through atmospheric fog is the most immediate example.

Investigation of this problem dates at least from Sewell [21], the most detailed analyses being those of Epstein [22] and Epstein and Carhart [23]. Interestingly enough, this work explicitly neglects the effects of condensation or vaporization of the droplets. Oswatitsch [24], on the other hand, investigated some effects of condensation, neglecting the usual damping mechanisms. With the principles we have developed, it is not difficult to include all of the contributing factors.

When the wave length of the sound is large compared with the droplet spacing, then the continuum-like model we have described here is applicable. This viewpoint is substantially simpler than that of Epstein and Carhart, who consider the details of gas and heat flow in the neighborhood of each particle. Temkin and Dobbins [25] have, in fact, shown that for solid particles in gas, the continuum-like approach leads to results indistinguishable from Epstein and

Carhart. We shall now extend the theory of sound propagation to include the phase change effects.

The most usual examples of fogs consist of droplets, vapor and an inert gas. Frequently, the inert gas constitutes the larger part of the mass per unit volume. For the present calculation, we shall neglect the differences between thermodynamic properties of vapor and inert gas so that we may define gas state  $p$ ,  $\rho$ ,  $T$ , and an equation of state independent of its composition. The partial pressure of the vapor  $p_v$  and partial density  $\rho_v$  will enter in the formulation of exchange processes. Then all equations of continuity, motion and energy are identical with those formulated in Section 3. The differences arise in the Clausius-Clapeyron equation, which must now be written

$$\frac{p_v^e(T_p)}{p_v^e(T_0)} = \exp \left\{ \frac{h_l}{RT_0} \left( 1 - \frac{T_0}{T_p} \right) \right\} \quad (7.1)$$

and in the vapor production rate  $\mu_v$ . Because of the inert gas, we assume the vapor transfer between a droplet and its surroundings to be diffusion controlled, so that from (2.16), utilizing the characteristic time  $\tau_0$

$$\mu_v = \frac{-\rho_p}{\Sigma^2 \tau_D} \left\{ \frac{p_v - p_v^e(T_p)}{p} \right\}. \quad (7.2)$$

The partial pressure  $p_v$  of the vapor component occurs in these two relations and thus must be accounted for by an equation of continuity. Since this component is the one produced by vaporization, this equation is

$$\frac{\partial \rho_v}{\partial t} + \frac{\partial}{\partial x_i} (\rho_v u_i) = \mu_v \quad (7.3)$$

where  $\rho_v$  is the partial density of the vapor. Then by suitable combination with the equation of continuity for the complete gas mixture, we obtain

$$\frac{\partial}{\partial t} \left( \frac{\rho_v}{\rho} \right) + u_i \frac{\partial}{\partial x_i} \left( \frac{\rho_v}{\rho} \right) = \frac{1}{\rho} \left( 1 - \frac{\rho_v}{\rho} \right) \mu_v. \quad (7.4)$$

Since all gaseous components are assumed to have the same temperature  $\rho_v/\rho = p_v/p$  so that Eq. (4.4) holds also for  $(p_v/p)$  and will be employed to relate the vapor pressure to other variables in our problem. The remainder of the mathematical problem is described by Eqs. (3.1), (3.2), (3.5), (3.6), (3.11), and (3.13).

Consider oscillatory plane waves with propagation direction parallel to the  $x$ -axis; the amplitudes are sufficiently small that the conventional acoustical linearization is valid. Because the medium has no average velocity parallel to the  $x$ -axis, the linearized equations for the gas (gas plus vapor) phase are

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u}{\partial x} = -\frac{\kappa_p \rho_0}{\tau_D} \left( \frac{p_v - p_v^e}{p_0} \right) \quad (7.5) \quad \frac{c}{c_p} \frac{\partial T_p}{\partial x} = -\frac{1}{\tau_T} (T_p - T)$$

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\kappa_p \rho_0}{\tau_v} (u_p - u) \quad (7.6) \quad -\frac{1}{\tau_D} \left( \frac{p_v^e - p_v}{p_0} \right) T_0 \eta \quad (7.11)$$

$$\rho_0 c_p \frac{\partial T}{\partial t} = \frac{\partial p}{\partial t} + \frac{\kappa_p \rho_0 c_p}{\tau_T} (T_p - T) \quad (7.7)$$

where  $\kappa_p$  is the ratio of droplet mass to total gas mass per unit volume in the undisturbed state,  $\kappa_p = \rho_{p0}/\rho_0$ . In addition, the equation of state becomes, when linearized,

$$\frac{p}{p_0} = \frac{\rho}{\rho_0} + \frac{T}{T_0} \quad (7.8)$$

Similarly, for the droplet phase, the Eqs. (3.2), (3.6), and (3.13) become

$$\frac{\partial \rho_p}{\partial t} + \kappa_p \rho_0 \frac{\partial u}{\partial x} = \frac{\kappa_p \rho_0}{\tau_D} \left( \frac{p_v - p_v^e}{p_0} \right) \quad (7.9)$$

$$\frac{\partial u_p}{\partial t} = -\frac{1}{\tau_v} (u_p - u) \quad (7.10)$$

where  $\eta = h_l/c_p T_0$  as before. The equilibrium vapor pressure  $p_v^e(T_p)$  follows from the Clausius-Clapeyron equation as

$$\frac{p_v^e}{p_v^e(T_0)} = \frac{\gamma}{\gamma - 1} \eta \frac{T_p}{T_0} \quad (7.12)$$

Finally, the continuity Eq. (7.4) for the vapor component may be linearized to give

$$\frac{\partial}{\partial t} \left( \frac{p_v}{p_0} - \kappa_v \frac{p}{p_0} \right) = (1 - \kappa_v) \kappa_p \frac{1}{\tau_D} \left\{ \frac{p_v^e - p_v}{p_0} \right\} \quad (7.13)$$

where  $\kappa_v \equiv \rho_{v0}/\rho_0$  denotes the ratio of vapor mass to gas mass per unit volume.

The acoustic problem for the condensing, two-phase flow is described by Eqs. (7.5), (7.6), (7.7), (7.10), (7.11), and (7.13). In all, six relationships are involved, where only five were required when we treated stationary waves in a pure medium. This increase results from

Table 4

Square of wave speed	Value	Relaxation process		
		$\mathcal{T}_v$	$\mathcal{T}_T$	$\mathcal{T}_D$
$a^2$	$\gamma p_0/\rho_0$	$\infty$	$\infty$	$\infty$
$a_v^2$	$\frac{1}{1 + \kappa_p} a^2$	0	$\infty$	$\infty$
$a_T^2$	$\frac{c_p + c \kappa_p}{c_p + \gamma c \kappa_p} a^2$	$\infty$	0	$\infty$
$a_D^2$	$\frac{(1 - \kappa_v) \kappa_p + \frac{c_p}{c} \kappa_v \frac{\gamma}{\gamma - 1} \eta^2}{\gamma \kappa_p \kappa_v + (1 - \kappa_v) \kappa_p + \frac{c_p}{c} \kappa_v \frac{\gamma}{\gamma - 1} \eta^2} a^2$	$\infty$	$\infty$	0
$a_{vT}^2$	$\left( \frac{a_v^2}{a^2} \right) \left( \frac{a_T^2}{a^2} \right) a^2$	0	0	$\infty$
$a_{vD}^2$	$\left( \frac{a_v^2}{a^2} \right) \left( \frac{a_D^2}{a^2} \right) a^2$	0	$\infty$	0
$a_{TD}^2$	$\frac{(1 - \kappa_v) \left( 1 + \frac{c}{c_p} \kappa_p \right) + \kappa_v \frac{\gamma}{\gamma - 1} \eta^2}{(1 - \kappa_v) \left( 1 + \frac{c}{c_p} \kappa_p \right) + \gamma \kappa_v \left( 1 + \frac{c}{c_p} \kappa_p \right) - 2\gamma \kappa_v \eta + \gamma \kappa_v \frac{\gamma}{\gamma - 1} \eta^2} a^2$	$\infty$	0	0
$a_{vTD}^2$	$\left( \frac{a_v^2}{a^2} \right) \left( \frac{a_{TD}^2}{a^2} \right) a^2$	0	0	0

the necessity to differentiate between the mass fraction of vapor and inert gas in the present problem.

The reduction to a single differential equation for any of the dependent variables proceeds in the same general manner as the argument of Section 6. We define a group of 8 sound speeds, 7 of them corresponding to constraints of the characteristic times in a similar manner to the constraints in characteristic length considered previously. Table 4 gives these wave speeds in detail, together with the constraints on the relaxation processes which would yield them physically. There is, of course, a strong similarity with those for the pure substance, given in Table 3. In fact, if the vapor mass fraction becomes unity, that is, if the mass fraction  $1 - \kappa_v$  of inert gas vanishes, the two sets become identical. Notice that there is no singular behavior of these quantities as the inert gas  $1 - \kappa_v$  becomes small and vanishes from the system. Thus, it is a perfectly reasonable approximation to set  $\kappa_v = 1$  in the wave speed. It must be remembered, however, that the physical processes entering into the relaxation time for phase change are altered in principle when  $\kappa_v = 1$ . When the inert component in the gas mixture vanishes, the phase change rate is no longer governed by diffusion but rather by the kinetic condensation rate. This difference was discussed in Section 2. It is to be noted also that when both the droplet mass fraction  $\kappa_p$  and the vapor fraction  $\kappa_v$  approach zero, all wave speeds approach the frozen speed "a"; this is to be expected, inasmuch as the system then consists only of the inert gaseous component.

The differential equation for any of the dependent variables, for example, the pressure perturbation  $p/p_0$ , may be written in a form similar to that we have used for the steady-state wave equation

$$\begin{aligned} & \left( \frac{\partial}{\partial t} \right)^3 \square^2 \left( \frac{p}{p_0} \right) + \left( \frac{\partial}{\partial t} \right)^2 \left\{ \frac{1}{\mathcal{T}_r} \square_v^2 \left( \frac{p}{p_0} \right) \right. \\ & + \frac{1}{\mathcal{T}_r} \square_r^2 \left( \frac{p}{p_0} \right) + \frac{1}{\mathcal{T}_D} \square_b^2 \left( \frac{p}{p_0} \right) \Big\} \\ & + \frac{\partial}{\partial t} \left\{ \frac{1}{\mathcal{T}_r \mathcal{T}_T} \square_{vT}^2 \frac{p}{p_0} + \frac{1}{\mathcal{T}_v \mathcal{T}_D} \square_{vD}^2 \left( \frac{p}{p_0} \right) \right. \\ & + \frac{\alpha}{\mathcal{T}_r \mathcal{T}_D} \square_{TD}^2 \left( \frac{p}{p_0} \right) \Big\} \\ & + \frac{\alpha}{\mathcal{T}_r \mathcal{T}_D \mathcal{T}_T} \square_{vTD}^2 \left( \frac{p}{p_0} \right) = 0 \end{aligned} \quad (7.14)$$

The modified relaxation time constants are defined as

$$\left. \begin{aligned} \frac{1}{\mathcal{T}_v} &= \frac{1}{\tau_v} \\ \frac{1}{\mathcal{T}_T} &= \frac{(c_p/c) + \kappa_p}{\tau_T} \\ \frac{1}{\mathcal{T}_D} &= \frac{(1 - \kappa_v) \kappa_p + (c_p/c) \kappa_v [\gamma/(\gamma - 1)] \eta^2}{\tau_D} \end{aligned} \right\} \quad (7.15)$$

the parameter  $\alpha$  as

$$\alpha \equiv \frac{(c/c_p) \kappa_p}{1 + (c/c_p) \kappa_p} \times \left[ 1 + \frac{1}{(1 - \kappa_v)(c/c_p) \kappa_p + \kappa_v [\gamma/(\gamma - 1)] \eta^2} \right] \quad (7.16)$$

and the wave operators are now defined, for example,

$$\square_{vD}^2 \equiv \frac{\partial^2}{a_{vD}^2 \partial t^2} - \frac{\partial^2}{\partial x^2} \quad (7.17)$$

each utilizing the constrained wave propagation speed indicated by the subscripts. It is of particular interest to note that, although the relaxation time constants behave in a perfectly normal manner as the vapor and liquid concentrations are varied, the parameter  $\alpha$  vanishes as the droplet concentration approaches zero. In physical terms, this has the effect of increasing the two relaxation time constants  $\tau_D \tau_T$  and  $\tau_v \tau_T \tau_D$  to very large values for small concentrations of liquid. The origin and details of this phenomenon were examined in Section 5. Its relevance to acoustic damping was first pointed out to the author by Professor R. A. Dobbins; it appears that Oswatitsch [24] recognized the physical consequence of small liquid mass.

To examine the spatial attenuation of plane acoustic waves caused by the momentum, heat, and mass exchange between vapor and liquid components, let the perturbation pressure be given in the form

$$\frac{p}{p_0} \sim \exp \left\{ i \left( \frac{Kx}{a} - \omega t \right) \right\} \quad (7.18)$$

where  $K \equiv K_1 + iK_2$  is complex. We are interested in the imaginary part  $K_2$  that concerns the spatial damping. The similar formulation may be effected for any of the dependent variables. The relationship between  $K$  and the angular frequency follows from the requirement that the determinant, formed by substituting (7.19) into the six relevant, homogeneous differential equations, must vanish. This determinant is explicitly



$iK$	0	$i\omega$	0	$-i\omega$	$-\frac{\kappa_p}{\tau_D}$	$= 0 \quad (7.19)$
$-i\omega + \frac{\kappa_p}{\tau_v}$	$-\frac{\kappa_p}{\tau_v}$	0	0	$\frac{iK}{\gamma}$	0	
0	0	$-i\omega + \frac{\kappa_p}{\tau_T}$	$-\frac{\kappa_p}{\tau_T}$	$i\omega \left( \frac{\gamma - 1}{\gamma} \right)$	0	
$-\frac{1}{\tau_v}$	$-i\omega + \frac{1}{\tau_v}$	0	0	0	0	
0	0	$-\frac{1}{\tau_T}$	$-i\omega \frac{c}{c_p} + \frac{1}{\tau_T}$	0	$\frac{\eta}{\tau_D}$	
0	0	0	$i\omega \frac{\gamma}{\gamma - 1} \kappa_v \eta^2$	$-i\omega \kappa_v$	$-i\omega + \frac{(1 - \kappa_v)}{\tau_D} \kappa_p$	

Fortunately, this result may be factored rather easily (reference to Eq. (7.14) clarifies this possibility), with the result that

$$\left(\frac{K}{\omega}\right)^2 = \frac{(Z_1 - i\omega)(Z_2 - i\omega)(Z_3 - i\omega)}{(\Omega_1 - i\omega)(\Omega_2 - i\omega)(\Omega_3 - i\omega)} \quad (7.20)$$

where

$$\begin{aligned} \Omega_1 &= \frac{1}{\mathcal{T}_v} \\ \Omega_2, \Omega_3 &= -\frac{1}{2} \left( \frac{1}{\mathcal{T}_T} + \frac{1}{\mathcal{T}_D} \right) \\ &\quad \pm \frac{1}{2} \sqrt{\left[ \left( \frac{1}{\mathcal{T}_T} + \frac{1}{\mathcal{T}_D} \right)^2 - \frac{4\alpha}{\mathcal{T}_T \mathcal{T}_D} \right]} \end{aligned} \quad (7.21)$$

and

$$\begin{aligned} Z_1 &= \frac{a^2}{a_v^2 \mathcal{T}_v} \\ Z_2, Z_3 &= -\frac{1}{2} \left( \frac{a^2}{a_T^2 \mathcal{T}_T} + \frac{a^2}{a_D^2 \mathcal{T}_D} \right) \\ &\quad \pm \frac{1}{2} \sqrt{\left[ \left( \frac{a^2}{a_T^2 \mathcal{T}_T} + \frac{a^2}{a_D^2 \mathcal{T}_D} \right)^2 - \frac{4\alpha a^2}{a_T^2 \mathcal{T}_T \mathcal{T}_D} \right]}. \end{aligned} \quad (7.22)$$

The radicals in (7.21), (7.22) may be shown to be real in all circumstances.

Generally speaking, the imaginary part of  $K$ , computed from (7.20), shows three maxima over the range of  $\omega$ , corresponding to the three general time constants of the problem. The expressions are too complex to be very informative, but specific numerical calculations give a good physical idea of the response. Figure 3 shows one of the calculations performed by Wooten [26] for the case where  $\tau_v = \tau_T = \tau_D$ ;  $\kappa_v = 0.5$ ,  $\eta = 10$ ,  $c_p/c = 1$ , and  $\gamma = 1.6$ , a situation

roughly applicable to an alkali metal vapor and liquid droplet mixture diluted 50 per cent by argon. The damping coefficient is shown in solid lines for three values of droplet loading  $\kappa_p$ . The broken lines give the corresponding curves calculated without mass transfer.

Because the relaxation times  $\tau_v$ ,  $\tau_T$ ,  $\tau_D$  were chosen equal, distinct maxima corresponding to each of them do not appear. As the mass fraction of liquid decreases, the maximum value of the damping coefficient drops and moves toward lower frequencies. The behavior is quite different for non-condensing droplets (broken line); the maximum remains at roughly the same frequency and decreases much more rapidly than do the results for condensing droplets. For low values of the liquid mass fraction, the phase-exchange process clearly dominates the damping and, upon closer examination of Fig. 3, the peak amplitude does not seem to decrease appreciably as  $\kappa_p$  continues to drop. It is most significant to realize, therefore, that phase exchange damping can be large even for very small mass fractions of liquid phase. We shall return to examine this phenomenon analytically later.

Figure 4 has been calculated to show the effects of the latent heat parameter  $[\gamma/(\gamma - 1)] \eta = h_l/RT_0$  upon the damping, holding all other parameters constant. For the liquid and vapor fractions chosen, all damping effects accumulate at  $\omega\tau_v = 1$ ; the frequency shift associated with condensation is not large. Moreover, variation of  $h_l/RT_0$  over its interesting range causes only a doubling of the damping coefficient. Two factors contribute to this: (i) Because the constants are so chosen that momentum, heat transfer, and phase exchange damping accumulate at the same frequency, the change in phase exchange damping

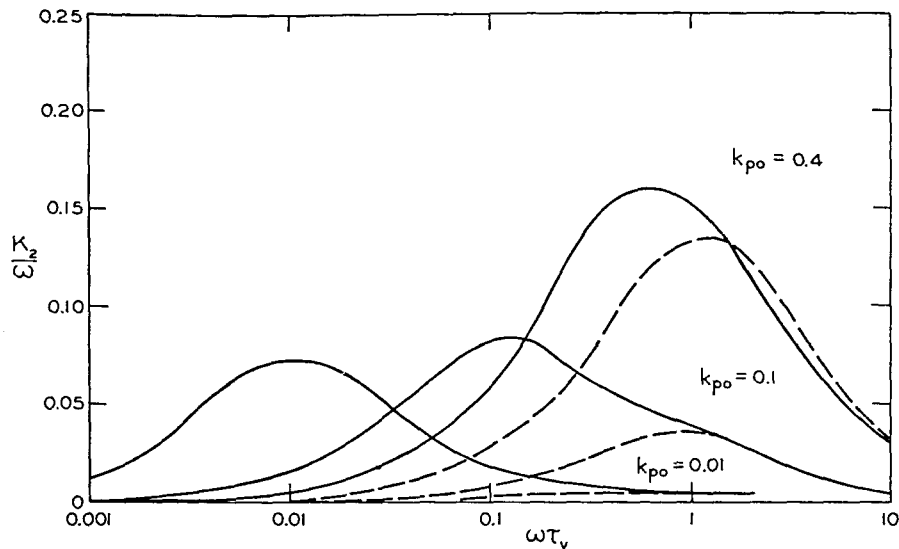


FIG. 3. Effect of phase exchange upon the acoustic damping caused by liquid droplets. Various liquid fractions,  $\kappa_v = 0.5$ ,  $hl/RT_0 = 10$ ,  $\gamma = 1.6$ ,  $\mathcal{T}_v = \mathcal{T}_T = \mathcal{T}_D$ .

must be compared with a large residual viscous and heat transfer damping. If the liquid fraction had been low enough to shift the maximum response frequency for phase exchange damping to low frequencies, the effects would have been proportionately lower. (ii) The increase in latent heat parameter has some compensating properties. The damping effect increases with latent heat, but the actual mass exchange, which also contributes to the damping, decreases.

Finally, Fig. 5 shows a case where small liquid

fraction has effectively separated the phase exchange damping response from that due to momentum and heat transfer. Moreover, the mass fraction of vapor,  $\kappa_v$ , is small enough so that the phase exchange damping does not dominate completely as it did in Fig. 3. The parameters are in the appropriate general range for atmospheric fog. It demonstrates clearly the existence of two distinct absorbing bands in such an atmosphere. The broken line again indicates the effect of neglecting phase exchange in the calculations.

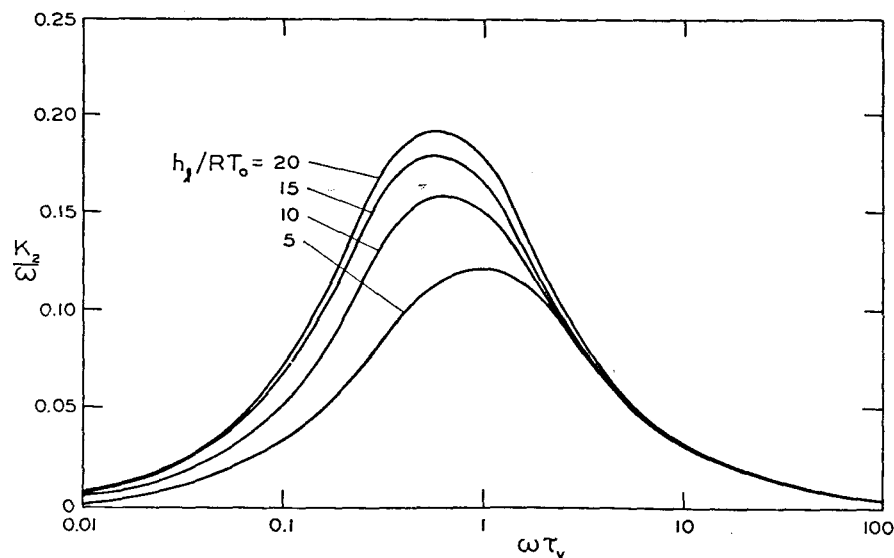


FIG. 4. Effect of latent heat parameter upon phase-exchange acoustic damping.  $\kappa_v = 0.5$ ,  $\kappa_p = 0.4$ ,  $\gamma = 1.6$ ,  $\mathcal{T}_v = \mathcal{T}_T = \mathcal{T}_D$ .

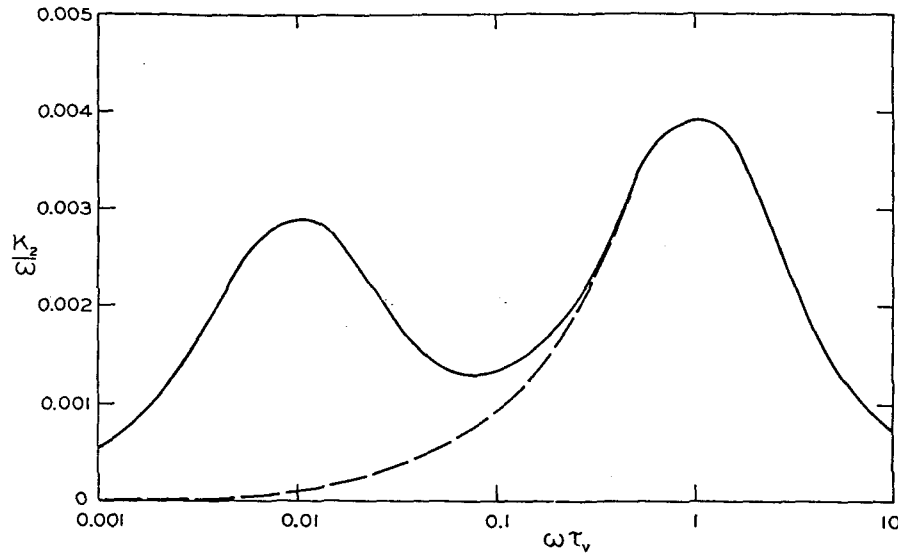


FIG. 5. Acoustic damping at very low liquid and vapor fractions. Broken line indicates damping without phase exchange,  $\kappa_p = \kappa_v = 0.01$ ,  $h_l/RT_0 = 5.0$ ,  $\gamma = 1.6$ ,  $\mathcal{T}_v = \mathcal{T}_T = \mathcal{T}_R$ .

The effects of small mass fraction of liquid upon the damping properties of fogs, that have been observed in the calculations, may be investigated analytically because the absorption peaks associated with phase change are remote from those resulting from other damping mechanisms. Then the frequencies we are investigating are so low that we may consider the viscous and heat-transfer processes nearly in equilibrium. This may be accomplished analytically by letting  $\tau_v, \tau_D \rightarrow 0$  in Eq. (7.14), the general acoustic relation. This yields the operator

$$\frac{\partial}{\partial t} \left( \frac{\partial^2}{a_{vT}^2 \partial t^2} - \frac{\partial^2}{\partial x_2^2} \right) + \left( \frac{\kappa_p}{\tau_D} \right) \times \left[ \frac{(1 - \kappa_v) \left( \frac{c_p}{c} + \kappa_p \right) + \frac{c_p}{c} \kappa_v \frac{\gamma}{\gamma - 1} \eta^2}{\frac{c_p}{c} + \kappa_p} \right] \times \left( \frac{\partial^2}{a_{vTD}^2 \partial t^2} - \frac{\partial^2}{\partial x_2^2} \right)$$

so that substitution of the pressure fluctuation according to (7.18), the value of the complex quantity  $K$ , satisfies the relationship.

$$\left( \frac{K}{\omega} \right)^2 = \frac{a^2}{a_{vT}^2} \left\{ \frac{-i\omega + \Omega_D \frac{a_{vT}^2}{a_{vTD}^2}}{-i\omega + \Omega_D} \right\} \quad (7.23)$$

where

$$\Omega_D = \frac{\kappa_p}{\tau_D} \left[ \frac{(1 - \kappa_v) \left( \frac{c_p}{c} + \kappa_p \right) + \frac{c_p}{c} \kappa_v \frac{\gamma}{\gamma - 1} \eta^2}{\frac{c_p}{c} + \kappa_p} \right] \quad (7.24)$$

Now for small values of liquid and vapor concentration in an inert gas,  $\kappa_p \ll 1$  and  $\kappa_v \ll 1$ , the situation in which we are interested at the moment, critical examination of Table 4 reveals that  $a_{vT}^2$  and  $a_{vTD}^2$  differ only in the order of these small quantities. Therefore, we may write the imaginary part of  $K$ , associated with the acoustic damping, as

$$\frac{K_2}{\omega} \cong \frac{1}{2} \frac{a}{a_{vT}} \frac{\frac{\omega}{\Omega_D}}{1 + \left( \frac{\omega}{\Omega_D} \right)^2} \left( \frac{a_{vT}^2 - a_{vTD}^2}{a^2} \right). \quad (7.25)$$

But utilizing the values of the appropriate constrained propagation speeds, it follows that

$$\frac{a_{vT}^2 - a_{vTD}^2}{a^2} \cong \gamma \kappa_v (\eta - 1)^2 \quad (7.26)$$

so that

$$\frac{K_2}{\omega} \cong \frac{\gamma}{2} \kappa_v (\eta - 1)^2 \frac{\frac{\omega}{\Omega_D}}{1 + \left( \frac{\omega}{\Omega_D} \right)^2}. \quad (7.27)$$

It is simple to show that this has a maximum at  $\omega = \Omega_D$ , that is,  $\omega^*$ , the frequency for maximum damping, is

$$\omega^* \cong \frac{\kappa_p}{\tau_D} \quad (7.28)$$

under the same approximation regarding  $\kappa_v$  and  $\kappa_p$ . The value of the maximum damping coefficient is therefore

$$\left(\frac{K_2}{\omega}\right)_{\max} = \frac{\gamma}{4} \kappa_v (\eta - 1)^2. \quad (7.29)$$

Equations (7.28) and (7.29) exhibit all of the important characteristics concerning phase exchange damping shown in Figs. 3–5. In the first place, the frequency for maximum damping decreases proportionately to the liquid mass fraction  $\kappa_p$ , as indicated by Eq. (7.28). This is shown clearly by the calculations shown in Fig. 3. Moreover, the peak value of the damping does not vary with liquid mass fraction, and this also is reflected in Fig. 3 for the lower values of  $\kappa_p$ . On the other hand, the general trend of increased damping with increased values of the latent heat parameter  $\eta$ , indicated in Fig. 4, is suggested by Eq. (7.29). However, because the values of  $\kappa_p$  and  $\kappa_v$  are not really small for those particular calculations, the magnitude of the variation is not reproduced. Equation (7.29) predicts a rather strong increase of damping and latent heat for small mass fractions of liquid and vapor. Figure 5 shows well the effects of decreasing vapor concentration upon the peak damping, for the values of damping indicated in this curve are small compared to those shown in Fig. 3 and 4 in proportion to the value of  $\kappa_v$ , as indicated by Eq. (7.29).

### 8. Concluding Remarks

It has been the general aim of this paper to show the manner in which the physical process of phase transition may be incorporated into the general formulation of continuum mechanics. In the process of carrying out this demonstration, several areas of technological interest have been touched upon, and the manner in which the analysis is adapted to them is illustrated. Some of the restrictions placed upon the formulation were a matter of convenience and may be removed easily. The linear transfer laws such as Stokes flow, the constancy of gas properties and latent heat, the single droplet radius, and other such items may be removed through rather straightforward extensions, but at the expense of much complexity and loss of clarity in the results. Other

restrictions have been imposed as a result of insufficient knowledge. Among the most serious of these are the break-up of droplets due to shear and acceleration, the division and agglomeration of droplets resulting from collisions, and some degree of uncertainty regarding the effects of mass-transfer rate upon the drag and heat-transfer laws. In some cases, such as strong shock waves or problems involving impingement upon surfaces, this lack of basic information severely limits the extension of the theory. This situation emphasizes the need for well-conceived, reliable experiments. Although they are difficult, the full technological understanding of two-phase flow requires this experimental background regardless of its difficulty.

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**Résumé**—Quelques problèmes gazeux dynamiques dans l'écoulement des vapeurs en condensation. Le problème générale de l'écoulement d'une vapeur humide, avec ou sans un agent diluant inerte est formulé en partant de la supposition que la phase liquide est finement divisée et dispersée dans tout le composant gazeux en gouttelettes dont les rayons sont presque constants dans toute région locale. Les processus de transfert d'impulsion, transfert de chaleur entre les phases sont supposés avoir lieu conformément à la loi de Stokes et le nombre d'unité de Nusselt respectivement. Le processus de transfert de la masse est traité comme diffusion contrôlée en présence d'un agent diluant inerte et la cinétique est contrôlée l'espace de deux phases d'une substance pure.

La compréhension physique de tels problèmes, contrastant avec ceux de la dynamique conventionnelle des gaz, repose largement sur la rôle joué par les temps de détente ou les longueurs d'équilibre associées à ces trois processus. En conséquence, les processus de détente aussi bien simples que couplés sont examinés avec un certain soin au moyen d'exemples spécifiques. Ensuite, le problème de l'écoulement proche d'équilibre dans une tuyère avec un changement de phases est résolu au moyen de l'approximation par petites valeurs de glissement.

La structure d'un choc normal dans une substance pure est examinée et révèle trois zones plutôt distinctes: le choc gazeux dynamique, la zone de détente de la vapeur, la zone d'équilibre thermique et de la vitesse. L'écoulement régulier à trois dimensions du milieu de condensation à deux phases est formulé conformément à la théorie des perturbations du premier degré et la structure des ondes dans un écoulement supersonique d'une telle nature est examiné. Finalement, l'atténuation du son dans les brouillards est formulée et résolue en tenant compte des importants effets du changement de phases ainsi que de l'amortissement visqueux et du transfert de chaleur qui ont été compris dans les analyses précédentes.

**Резюме**—Некоторые Газодинамические Проблемы в Потоке Конденсированных Паров. (I) Формулируется общая проблема потока влажного пара с инертным разбавителем или без него с предположением, что жидкая фаза в конце делится и рассеивается по всему газообразному компоненту каплями, радиусы которых почти постоянны в любом местном районе. Допускается, что процессы передачи импульсов, теплопередача между фазами, происходят соответственно закону Стокса и числу единства Нуссельта. Процесс передачи массы рассматривается, как диффузия, управляемая в присутствии инертного разбавителя и кинетически управляется вследствие двух фаз чистого вещества.

Физическое понимание таких проблем, в противоположность динамик обыкновенного газа, основывается, главным образом, на роли времени релаксации или длины установления равновесия, связанных с этими тремя процессами. Поэтому, как простые, так и спаренные процессы релаксации довольно тщательно рассматриваются специфическими примерами. Впоследствии проблема почти уравновешенного потока в сопле с изменением фазы разрешается в приближении небольшого скольжения. Исследуется структура нормального удара в чистой субстанции и обнаруживается три довольно различных зоны: газодинамический удар, зона релаксации пара и зона термического уравновешивания и уравновешивания скорости. Формулируется трехмерный установившийся поток двухфазного конденсированного континуума по первому порядку теории возмущения и исследуется структура волн в таком сверхзвуковом потоке. Наконец, формулируется и разрешается поглощение звука в тумане, учитывая важный эффект изменения фазы также, как и вязкое глушение и теплопередача, которые включены в предыдущие анализы.